Triangulations and the Euler characteristic

Let S be a compact connected surface. In this note, we shall consider S as a *topological surface*, meaning a Hausdorff topological space such that each point p in S has an open neighbourhood $U = U_p$ homeomorphic to an open disc in $\mathbb{R}^2$.

There is an important topological invariant called the Euler characteristic. In order to define it, we shall need a concept of *triangulation.* Here is an outline, assuming some statements without proofs.

Informally, a triangulation $\mathcal{T}$ is cutting a surface S into a finite number of 'polygonal' regions, called faces, by smooth non-self-intersecting arcs, called edges, joined at vertices (so that a triangulated surface looks like a 'topological polyhedron'). More precisely, an *edge* of $\mathcal{T}$ is a homeomorphic image in S of the interval $[0, 1] \subset \mathbb{R}$ and the images of 0 and 1 are *vertices* of $\mathcal{T}$. The complement of the edges of $\mathcal{T}$ in S consists of (finitely many) connected components; each one is required to be homeomorphic to an open disc. The *faces* of $\mathcal{T}$ are the closures of these components. In addition, one requires the following properties:

- any two faces share only one edge, if at all; each edge belongs to the boundaries of exactly two faces;
- two edges meet only in one common end-point (vertex), if at all;
- any vertex has a neighbourhood homeomorphic to an open disc with edges corresponding to rays from the centre to the boundary of that disc. Any two distinct sectors cut out by these rays correspond to distinct faces of $\mathcal{T}$. (Consequently, at least 3 edges meet at each vertex.)

Remark. In the literature, there are some variations on what is allowed or disallowed in a triangulation. However, these variations will not be important for us as they lead to the same Euler characteristic.

Let $V(\mathcal{T}), E(\mathcal{T}), F(\mathcal{T})$ denote the number of vertices, edges, and faces of $\mathcal{T}$. A remarkable fact is that the quantity

$$\chi(S, \mathcal{T}) = V(\mathcal{T}) - E(\mathcal{T}) + F(\mathcal{T})$$

is independent of the choice of a triangulation of S. Therefore it can be written as $\chi(S)$ and is called the *Euler characteristic* of S. Euler characteristic also appears in the Algebraic Topology course and the Riemann Surfaces course.

We shall assume without proof the following topological result.

Theorem. *Every compact surface S in $\mathbb{R}^n$ has a triangulation.*

What really matters here is whether a surface S is a so-called 'second countable' topological space. By definition, second countable means that there is a countable family of open neighbourhoods $U_n \subset S$, $n = 1, 2, ...$, so that every open subset of S can be obtained as a union of some U_n's.

The Euclidean space $\mathbb{R}^n$, for each n, is second countable as one can choose the family of all the open balls with rational radii and with centres whose coordinates are rational numbers. Surfaces in $\mathbb{R}^n$, with the subset topology, are thus second countable.

Compact connected *orientable* surfaces (without boundary) are classified up to a homeomorphism by their Euler characteristic χ (or, equivalently, by the genus). The *genus* $g(S)$ is related to $\chi(S)$ by $\chi(S) = 2 - 2g(S)$ and $g(S)$ can be visualized as 'the number of handles that one needs to attach to the sphere in order to obtain S'. Thus $g(S) \geq 0$ and $\chi(S) \leq 2$.

For example, $\chi(S^2) = 2$ and $g(S^2) = 0$; we can use a triangulation with $V = F = 4$ and $E = 6$ induced by putting a regular tetrahedron inside the sphere and projecting from a point inside the tetrahedron. The torus has $\chi = 0$ and $g = 1$, another example is shown in Figure 1.

> You may notice from the above that $\chi(S)$ is even for each compact orientable surface S. An example of topological surface with an odd χ is the quotient $S^2/\pm 1$ by the antipodal map (if the antipodal map preserves the triangulation $\mathcal{T}$ of S^2 then the quotient has a triangulation with exactly half of everything $\mathcal{T}$ has, hence $\chi = 1$). This is the projective plane $\mathbb{R}P^2$. It is not orientable and can be realized as a manifold in $\mathbb{R}^n$ for $n \geq 4$, but not in $\mathbb{R}^3$ (there is an immersion of $\mathbb{R}P^2$ in $\mathbb{R}^3$ with self-intersection known as a 'cross-cap').

The notion of triangulation extends to compact surfaces S *with boundary*. Then ∂S is necessarily a disjoint union of circles S^1 and we require that each S^1 is a union of edges (it follows from the previous conditions that each S^1 will contain at least 2 vertices). E.g. a closed disc has $\chi = 1$ (a sphere minus one face).

A triangulation can be refined, if necessary, by performing sufficiently many times a *barycentric subdivision*: mark a point, a new vertex, in the interior of a face and further new vertices in the middle of each edge of this face. Join the new vertex in the interior by new edges with those in the middle of the edges and also with all the 'old' vertices on the boundary of this face.

In this way, one can achieve triangulations with very small faces and with each face diffeomorphic to Euclidean triangle. Furthermore, the edges may be chosen to be arcs of geodesics.

Fig. 1. Triangulating a 'topological pretzel': $\chi = 24 - 44 + 18 = -2, \;\; g = 2$.