Term by term integration and differentiation

Sometimes the calculus one needs to do involves functions which cannot be defined in a traditional way by a formula, but only in terms of convergent series of 'elementary' functions. This then poses a question:
When is the formal term by term integration or differentiation of a series of functions valid (i.e. will give the same result as the integration or differentiation applied directly to the sum of the series)?

Earlier we proved the following.

Theorem 1. *Suppose that $f_n : [a, b] \to \mathbb{R}$, for each $k = 1, 2, \dots,$ is integrable on $[a, b]$ and that $f_n(x) \to f(x)$ uniformly on $[a, b]$ as $n \to \infty$.*

Then the limit function $f(x)$ is integrable on $[a, b]$ and $\int_a^b f(x)dx = \lim_{n\to\infty} \int_a^b f_n(x)dx$.

This can be easily converted into a version for the series.

Theorem 1$'$ (Term by term integration)**.** *Suppose that $u_k : [a, b] \to \mathbb{R}$, for each $k = 1, 2, \dots,$ is integrable on $[a, b]$ and $\sum_{k=1}^{\infty} u_k(x)$ converges uniformly on $[a, b]$.*

Then the sum $f(x) = \sum_{k=1}^{\infty} u_k(x)$ is integrable on $[a, b]$ and $\int_a^b f(x)dx = \sum_{k=1}^{\infty} \int_a^b u_k(x)$.

Proof. Put $s_n(x) = \sum_{k=1}^{n} u_k(x)$ and apply Theorem 1 to the sequence (s_n). □

Thus uniformly convergent series can be integrated term by term.

What about term by term differentiation? Here the situation is somewhat less elegant than with integration and there is a good reason for that.

Very *informally*, the integration is a 'bounded' operation whereas the differentiation is not. If $u(x)$ is 'small', say $|u(x)| < \varepsilon$ for each x, then the integral $|\int_0^1 u(x)dx| < \varepsilon$ is also 'small', but the derivative $|u'(x)|$ may be arbitrary 'large', consider e.g. $u(x) = \varepsilon \sin(x/\varepsilon^2)$ and let $\varepsilon \to 0$.

The next theorem is essentially a result for integration in disguise (as you will see from the proof). We can assume very little on the initial series, but the term by term differentiated series must satisfy a strong condition of being uniformly convergent.

Theorem 2 (Term by term differentiation)**.** *Suppose that $u_k : [a, b] \to \mathbb{R}$, for each $k = 1, 2, \dots,$ has continuous derivative on $[a, b]$ (at the end-points a and b this means one-sided derivative). Suppose further that*
(i) the series $\sum_{k=1}^{\infty} u_k(x_0)$ converges at some point $x_0 \in [a, b]$ and
(ii) the series of derivatives $\sum_{k=1}^{\infty} u'_k(x)$ converges uniformly on $[a, b]$, to $f(x) = \sum_{k=1}^{\infty} u'_k(x)$ say.

Then
(1) the series $\sum_{k=1}^{\infty} u_k(x)$ converges at every $x \in [a, b]$ and the sum $F(x) = \sum_{k=1}^{\infty} u_k(x)$ is differentiable with $F'(x) = f(x)$ for each $x \in [a, b]$;
(2) moreover, the convergence of $\sum_{k=1}^{\infty} u_k(x)$ to $F(x)$ is uniform on $[a, b]$.

Proof. (1) As each u'_k is continuous and $\sum_{k=1}^{\infty} u'_k$ is uniformly convergent, we have f continuous on $[a, b]$. [1] Therefore, each u'_k and f are integrable on $[a, b]$. Let $x \in [a, b]$. Applying Theorem 1$'$ to u'_k and f on $[x_0, x]$ (or on $[x, x_0]$ if $x < x_0$) we obtain

$$\sum_{k=1}^{\infty} \int_{x_0}^{x} u'_k(x)dx = \int_{x_0}^{x} f(x)dx. \tag{1}$$

[1] because we proved earlier that the limit of a uniformly convergent sequence of continuous functions is continuous

The left-hand side of (1) may be written as

$$\sum_{k=1}^{\infty}\int_{x_0}^{x} u_k'(x)dx = \lim_{n\to\infty}\sum_{k=1}^{n}\int_{x_0}^{x} u_k'(x)dx$$

$$= \lim_{n\to\infty}\sum_{k=1}^{n}\big(u_k(x)-u_k(x_0)\big) = \lim_{n\to\infty}\left(\sum_{k=1}^{n}u_k(x)-\sum_{k=1}^{n}u_k(x_0)\right) \quad (2)$$

The $\lim_{n\to\infty}\sum_{k=1}^{n}u_k(x_0)$ exists by hypothesis (i), therefore the series $F(x)=\lim_{n\to\infty}\sum_{k=1}^{n}u_k(x)$ converges for each $x\in[a,b]$. Thus a function $F:[a,b]\to\mathbb{R}$ is well-defined and from (1) and (2) we obtain

$$F(x)-F(x_0)=\int_{x_0}^{x} f(x)dx.$$

Differentiation in x gives $F'(x)=f(x)$ for $x\in[a,b]$ as f is continuous.

(2) Let $\varepsilon>0$. Then there is N_1 such that whenever $n\geq m\geq N_1$ we have

$$\left|\sum_{k=m}^{n}u_k(x_0)\right| < \frac{\varepsilon}{2},$$

applying the Cauchy criterion to the convergent series $\sum_{k=1}^{\infty}u_k(x_0)$. Also there is N_2 such that whenever $n\geq m\geq N_2$ we have

$$\left|\sum_{k=m}^{n}u_k'(x)\right| < \frac{\varepsilon}{2(b-a)}, \quad \text{for each } x\in[a,b],$$

applying the Cauchy criterion to the uniformly convergent series of functions $\sum_{k=1}^{\infty}u_k'(x)$.

Put $N=\max\{N_1,N_2\}$ and $g(x)=\sum_{k=m}^{n}u_k(x)$. Then for $x\in[a,b]$ we can write $g(x)-g(x_0)=g'(\xi)(x-x_0)$ with some $\xi\in[x_0,x]\subseteq[a,b]$ (or $\xi\in[x,x_0]\subseteq[a,b]$), using the Mean Value Theorem from Analysis I. We then obtain

$$|g(x)|\leq|g(x_0)|+|g(x)-g(x_0)|=|g(x_0)|+|g'(\xi)|\cdot|x-x_0|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2(b-a)}(b-a)=\varepsilon.$$

Thus for $n\geq m\geq N$ and for each $a\leq x\leq b$ we have $|\sum_{k=m}^{n}u_k(x)|\leq\varepsilon$ which is the Cauchy criterion for uniform convergence of $\sum_{k=1}^{\infty}u_k(x)$ as we had to prove. □

Once again, it is now easy to obtain a version of the result concerning differentiation for the sequences of functions.

Corollary 1. *Suppose that $f_n:[a,b]\to\mathbb{R}$, for each $n=1,2,\ldots$, has continuous derivative on $[a,b]$ and $f_n(x)\to f(x)$ point-wise for each $x\in[a,b]$. If $f_n'(x)\to\varphi(x)$ uniformly on $[a,b]$ then the function f is differentiable with $f'(x)=\varphi(x)$ for each $x\in[a,b]$. Moreover, $f_n'\to f$ uniformly on $[a,b]$.*

Proof. After putting $u_1=f_1$, $u_n=f_n-f_{n-1}$ for $n>1$, the result follows from Theorem 2. □

Remark 1. The condition that the derivatives u_k' in Theorem 2 (resp. f_n' in Corollary 1) are continuous can be dropped. The result then is still true but the proof is longer. You can find details in Körner's book (although he still assumes the continuity) or in Rudin's book.

Remark 2. One more remark for interest is that the condition of uniform continuity in Theorem 1, while certainly sufficient, is in fact too strong. It would suffice to assume e.g. that $|f_n(x)|\leq 1$ for all n and $x\in[a,b]$. The proof is not obvious and requires some more advanced theory of Lebesgues integral and measure; you can learn it in Part II.