DYNAMICAL SYSTEMS. EXAMPLES 1.

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Comments on and/or corrections to the questions on this sheet are always welcome, and may be e-mailed to me at `g.p.paternain@dpmms.cam.ac.uk`. Most of the examples in this sheet are taken from the text that I am following in lectures: *Introduction to dynamical systems,* by M. Brin & G. Stuck.

1. Show that the complement of a forward invariant set is backward invariant and viceversa. Show that if f is bijective, then an invariant set A satisfies $f^t(A) = A$ for all t. Show that this is false, in general, if f is not bijective.

2. Suppose (X, f) is a factor of (Y, g) by a semiconjugacy $\pi : Y \to X$. Show that if $y \in Y$ is periodic point, then $\pi(y) \in X$ is periodic. Give an example to show that the preimage of a periodic point does not necessarily contain a periodic point.

3. Let G be a topological group. Prove that if G has a minimal left translation, then G is abelian.

4. (a) Show that for any $k \in \mathbb{Z}$, there is a continuous semiconjugacy from R_α to $R_{k\alpha}$.
(b) Show that R_α and R_β are conjugate by a homeomorphism if and only if $\alpha = \pm\beta$ mod 1.

5. Let A be an $m \times m$ matrix with zeros and ones. Prove that:
(a) the number of fixed points of the shift in Σ_A (or Σ_A^+) is the trace of A;
(b) the number of allowed words of length $n + 1$ beginning with the symbol i and ending with j is the i, j-th entry of A^n; and
(c) the number of periodic points of the shift of period n in Σ_A (or Σ_A^+) is the trace of A^n.

6. Assume that all entries of some power of A are positive. Show that for the shift on Σ_A and Σ_A^+, periodic points are dense, and there are dense orbits.

7. Let X be a locally compact metric space and $f : X \to X$ a continuous map. Suppose p is an attracting fixed point for f. Show that there is a neighbourhood U of p such that the forward orbit of every point in U converges to p.

8. Let $f : \mathbb{R} \to \mathbb{R}$ be a C^1 map, and p be a fixed point. Show that if $|f'(p)| < 1$, then p is attracting, and if $|f'(p)| > 1$, then p is repelling.
Consider the quadratic family q_μ. Are 0 and $1 - 1/\mu$ attracting or repelling for $\mu = 1$? and for $\mu = 3$?

9. Show that the eigenvalues of a two-dimensional hyperbolic toral automorphism are irrational. Prove that the stable and unstable manifolds are dense.
Show that the number of periodic points of period n of a hyperbolic toral automorphism A is $\det(A^n - I)$.

10. Compute the Lyapunov exponents for the expanding map E_m and the solenoid.

11. The *Haar measure* of a compact metrisable topological group G is the unique probability measure μ defined on the Borel σ-algebra $\mathcal{B}$ of G such that $\mu(xE) = \mu(E)$ for all $x \in G$ and all $E \in \mathcal{B}$. Show that every non-empty open subset of G has non-zero Haar measure. Show that a continuous surjective endomorphism of G preserves Haar measure.

12. Prove that there exists a point in $x \in \mathbb{R}/\mathbb{Z}$ such that its ω-limit set with respect to the expanding map E_3 is the standard middle-third Cantor set K. In particular K is E_3-invariant and contains a dense orbit.

13. Show that the periodic points of any hyperbolic toral *endomorphism* F of $\mathbb{T}^n$ are dense. (Hint: consider a number p relatively prime to $\det F$ and look at F restricted to $p^{-1}\mathbb{Z}^n/\mathbb{Z}^n$.)