Exercise A.1. An alternative proposed proof of the chain rule runs as follows. We rewrite:

$$\frac{g(f(x)) - g(f(p))}{x-p} = \frac{g(f(x)) - g(f(p))}{f(x)-f(p)} \frac{f(x)-f(p)}{x-p},$$

and we claim:

$$\lim_{x \to p} \frac{g(f(x)) - g(f(p))}{f(x)-f(p)} = \lim_{y \to f(x)} \frac{g(y) - g(f(p))}{y - f(p)} = g'(f(p))$$

which together with:

$$\lim_{x \to p} \frac{f(x)-f(p)}{x-p} = f'(p)$$

gives the result. Why is this 'proof' not valid?

Exercise A.2. a) Suppose that $n \in \mathbb{N}$ and $f : \mathbb{R} \to \mathbb{R}$ is given by:

$$f(x) = x^n.$$

Show that f is differentiable at all points of $\mathbb{R}$ and $f'(x) = nx^{n-1}$.

b) Show that if $f : \mathbb{R} \to \mathbb{R}$ is any polynomial function of x, then f is differentiable at all points of $\mathbb{R}$.

Exercise A.3. Prove[3] that $f : \mathbb{R} \to \mathbb{R}$ is differentiable everywhere in $\mathbb{R}$, and find the derivative of:

a) $f(x) = \sin x$

b) $f(x) = \cos x$

c) $f(x) = \exp x$

d) $f(x) = e^{-x^2}$

Exercise A.4. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by:

$$f(x) := \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

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Questions marked $(*)$ are optional

[3]You may assume

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}, \quad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}, \quad \exp x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n.$$

a) Show that for $x \neq 0$, f is differentiable at x with derivative:

$$f'(x) = -\cos\frac{1}{x} + 2x\sin\frac{1}{x}.$$

[Hint: Use Theorems A.2, A.3]

b) Show that f is differentiable at $x = 0$ with derivative:

$$f'(0) = 0.$$

[Hint: directly write down $\frac{f(x)-f(0)}{x}$ and show that the limit as $x \to 0$ is 0.]

c) Show that the function $x \mapsto f'(x)$ is not continuous at $x = 0$.

[Hint: Consider the sequence $(x_i)_{x=1}^{\infty}$ with $x_i = (2\pi i)^{-1}$.]

Exercise A.5. Show that if the power series:

$$f(x) = \sum_{k=0}^{\infty} a_k x^k,$$

has radius of convergence $R > 0$, then the power series:

$$g(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}$$

also has radius of convergence R. Deduce that a power series is smooth inside its radius of convergence.

[Hint: Suppose $|x| = r < R$, and apply the comparison test to the second series, comparing with the series $\sum_{k=0}^{\infty} a_k s^k$ for some $r < s < R$,]

Exercise A.6. Suppose that $f, g : (a, b) \to \mathbb{R}$ are differentiable on (a, b) and satisfy:

$$f'(x) = g'(x), \qquad \text{for all } x \in (a, b).$$

Show that:

$$f(x) = g(x) + C.$$

[Hint: Consider $u(x) = f(x) - g(x)$.]

Exercise A.7. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable and satisfies:

$$f'(x) = f(x), \quad \text{for all } x \in \mathbb{R} \qquad f(0) = 1. \tag{6}$$

a) Show that the function:

$$f(x) = \exp x$$

is a solution of (6).

b) Let $g(x) = \frac{f(x)}{\exp(x)}$. Show that:

$$g'(x) = 0, \quad \text{for all } x \in \mathbb{R} \qquad g(0) = 1.$$

Conclude that $f(x) = \exp x$ is the *unique* solution of (6).

c) By considering $f(x) = \frac{1}{\exp(-x)}$ or otherwise, show that $\exp(-x) = \frac{1}{\exp x}$.

Exercise A.8. Let $p_k : \mathbb{R} \to \mathbb{R}$ be given by:

$$p_k(x) = a_k x^k + a_{k-1}x^{k-1} + \ldots + a_0$$

for some $a_i \in \mathbb{R}$. Show that there is a unique $f : \mathbb{R} \to \mathbb{R}$ which is differentiable everywhere in $\mathbb{R}$ and satisfies:

$$f'(x) = p_k, \qquad f(0) = 0,$$

and that f is a polynomial of degree $k + 1$.

[Hint: First find one solution, then show that it must be unique.]

Exercise A.9. Suppose that $f, g : (a, b) \to \mathbb{R}$ are $k-$times differentiable on (a, b) and satisfy:

$$f^{(k)}(x) = g^{(k)}(x), \qquad \text{for all } x \in (a, b).$$

Show that:

$$f(x) = g(x) + p_{k-1}(x).$$

for some polynomial p_{k-1} of order $k - 1$.

[Hint: Work by induction on k]

Exercise A.10. a) By using the power series or otherwise, show that for any $N \in \mathbb{N}^*$ and $x \geq 0$ that:

$$1 + \frac{x^N}{N!} \leq \exp x.$$

b) Deduce that for $y > 0$:

$$e^{-\frac{1}{y}} \leq \frac{1}{1 + \frac{1}{N!y^N}} \leq N!y^N.$$

c) Suppose p_k is a polynomial of degree k, and consider the function:

$$f(x) = \begin{cases} 0 & x \leq 0, \\ p_k\left(\frac{1}{x}\right) e^{-\frac{1}{x}} & x > 0. \end{cases}$$

Show that:

$$\lim_{y \to 0} f(y) = 0.$$

Exercise A.11 (*)**.** Consider the function:

$$f(x) = \begin{cases} 0 & |x| \geq 1, \\ e^{-\frac{1}{1-x^2}} & |x| < 1. \end{cases}$$

Show that f is smooth and non-negative on $\mathbb{R}$.

[Hint: Let

$$g(x) = \begin{cases} 0 & x \leq 0, \\ e^{-\frac{1}{x}} & x > 0. \end{cases}$$

and show that $f(x) = g\left[2(1+x)\right] g\left[2(1-x)\right]$*.]*