Chapter 3

Einstein's equations

3.1 Einstein's equations and matter models

We are now in a position to write down Einstein's equations for the gravitational field, which describe a 4−dimensional Lorentzian manifold $(\mathcal{M}, g)$:

$$Ric_g - \frac{1}{2}Rg + \Lambda g = T. \tag{3.1}$$

The left hand side of Einstein's equations involves terms constructed from the metric g. The first two terms are familiar from the discussion of the previous chapter. The third term on the left hand side is a constant multiple of the metric itself. Λ here is a parameter of the theory known as the *cosmological constant*. It was introduced by Einstein in order that the theory would admit solutions corresponding to a stationary universe. The discovery by Hubble that in fact galaxies are all moving apart caused Einstein to throw away this term, dismissing it as the "greatest blunder" of his life. Modern measurements of the Cosmic Microwave Background, and Type Ia Supernovae data suggest that the cosmological constant term should be present, and that $\Lambda > 0$.

The term on the right hand side, T, is the energy-momentum tensor of the matter present in the spacetime. It is a symmetric, divergence free tensor. In order to close the system of equations represented by (3.1), we have to specify some model for the matter present in the spacetime, which describes how the matter evolves in time. Possible matter models include:

1. **Vacuum**. For this we set $T \equiv 0$, so that there is no matter present in the spacetime. Bothe the Minkowski spacetime, and the Schwarzschild spacetime that we have already encountered are solutions of the vacuum Einstein equations with $\Lambda = 0$.

2. **Wave matter** The matter content is encoded in a single function ψ satisfying the wave equation:
$$\Box_g \psi = 0$$
where $\Box_g$ is the wave operator of the metric g. The energy-momentum tensor is then given in a local basis by:
$$T_{\mu\nu}[\psi] = \nabla_\mu \psi \nabla_\nu \psi - \frac{1}{2} g_{\mu\nu} \nabla_\sigma \psi \nabla^\sigma \psi.$$

3. **Electromagnetic**. Here the matter content is encoded in an antisymmetric $(0,2)-$tensor, F satisfying the Maxwell equations, which in a local basis take the form:
$$\nabla_\mu F^\mu{}_\nu = 0, \qquad \nabla_{[\mu} F_{\nu\sigma]} = 0,$$
where ∇ is the Levi-Civita connection of g. The energy-momentum tensor is then given in a local basis by:
$$T_{\mu\nu}[F] = F_{\mu\sigma}F_\nu{}^\sigma - \frac{1}{4}\eta_{\mu\nu}F_{\sigma\tau}F^{\sigma\tau}.$$

4. **Perfect fluid** A perfect fluid is described by a local velocity $U \in \mathfrak{X}(\mathcal{M})$, which is everywhere a unit timelike vector field, together with a pressure p and a density ρ. They satisfy the first law of thermodynamics:
$$U[\rho] + (\rho + p)\mathrm{div}_g U = 0,$$
and Euler's equation:
$$(\rho + p)\nabla_U U + \mathrm{grad}_g p + U[p]U = 0,$$
This gives a closed system once a relation, called an equation of state, $p = p(\rho)$ is specified. The energy momentum tensor is then given by
$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + pg_{\mu\nu}.$$

All of these matter models are used for various purposes in the study of relativity. Notice that in general, the equations of motion for the matter fields depend on the metric, and of course the metric evolves according to Einstein's equations. This was summed up by Wheeler as:

> "Space acts on matter, telling it how to move. In turn, matter reacts back on space, telling it how to curve."[1]

In general, we have a complicated system of nonlinear, hyperbolic PDEs for the 10 components of the metric and the matter fields. For most of the rest of the course, we will focus on the vacuum case. This allows us to consider some of the challenges of studying general relativity in a somewhat simpler setting.

3.2 The linearised Einstein equations

A starting point for the study of any nonlinear PDE is often to study the linearisation about a known solution. This usually results in a simpler problem, which can be attacked with standard methods. The knowledge one gains from studying the linearised problem can then be used to try and tackle the full, nonlinear, problem. We will consider the

[1] C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation*. W. H. Freeman, 1973.

problem of linearising the vacuum Einstein equations about the Minkowski space. In the vacuum case, with $\Lambda = 0$, the equations reduce to

$$Ric_g = 0.$$

Recall that Minkowski space is the manifold $\mathbb{R}^4$, with coordinates $(x^\mu)_{\mu=0,\ldots 3}$ and the metric given by:

$$\eta = \eta_{\mu\nu} dx^\mu dx^\nu = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2.$$

Now, since

$$\Box_\eta f = \eta^{\mu\nu} \frac{\partial^2 f}{\partial x^\mu \partial x^\nu},$$

the coordinates x^μ are wave coordinates for the Minkowski metric.

Let us suppose that we have a family of metrics ${}^{(s)}g$ defined on $\mathbb{R}^4$, with ${}^{(0)}g = \eta$ and which depend smoothly[2] on $s \in (-\epsilon, \epsilon)$. Suppose also that ${}^{(s)}g$ solves the Einstein equations, and that the coordinates x^μ are wave coordinates for ${}^{(s)}g$ for every $s \in (-\epsilon, \epsilon)$. This implies that

$$-\frac{1}{2} {}^{(s)}g^{\mu\alpha} \frac{\partial^2}{\partial x^\mu \partial x^\alpha} \left[{}^{(s)}g_{\sigma\nu} \right] + {}^{(s)}\Gamma_{\tau\lambda\nu} {}^{(s)}\Gamma^{\tau\lambda}{}_\sigma + {}^{(s)}\Gamma_{\tau\lambda\nu} {}^{(s)}\Gamma_\sigma{}^{\tau\lambda} + {}^{(s)}\Gamma_{\tau\lambda\sigma} {}^{(s)}\Gamma_\nu{}^{\tau\lambda} = 0$$

and

$$0 = {}^{(s)}\Gamma^{\alpha\mu}{}_\mu = {}^{(s)}g^{\alpha\tau} {}^{(s)}g^{\mu\nu} \left(\partial_\nu {}^{(s)}g_{\mu\tau} - \frac{1}{2} \partial_\tau {}^{(s)}g_{\mu\nu} \right).$$

To find the linearised Einstein equations, we differentiate these with respect to s, and then set $s = 0$. Recalling that ${}^{(0)}g_{\mu\nu} = \eta_{\mu\nu}$ and ${}^{(0)}\Gamma_{\tau\lambda\sigma} = 0$, we deduce that

$$0 = \Box_\eta \gamma_{\mu\nu}, \tag{3.2}$$

$$0 = \partial_\mu \gamma^\mu{}_\nu - \frac{1}{2} \partial_\nu \gamma_\sigma{}^\sigma, \tag{3.3}$$

where $\gamma_{\mu\nu} = \frac{d}{ds} \left[{}^{(s)}g_{\mu\nu} \right] \big|_{s=0}$, and indices are raised and lowered with η. Equation (3.2) simply says that the components of γ with respect to the canonical coordinates on Minkowski space each separately obey the wave equation. By the results of Chapter 1, a unique solution for $\gamma_{\mu\nu} \in C^\infty(\mathbb{R}^4)$ exists, provided we specify smooth initial data: $\gamma_{\mu\nu}|_{x^0=0}$ and $\partial_0 \gamma_{\mu\nu}|_{x^0=0}$. Can we simultaneously satisfy equation (3.3)? This represents a set of *constraints* that our solutions to (3.2) must satisfy. In order that the pair of equations (3.2), (3.3) admit any solutions at all, they must be compatible. That they are is a result of the following:

Lemma 3.1. *Suppose $\gamma_{\mu\nu}$ is a smooth solution of (3.2). Then (3.3) holds in $\mathbb{R}^4$ if, and only if:*

$$0 = \partial_\mu \gamma^\mu{}_\nu - \frac{1}{2} \partial_\nu \gamma_\sigma{}^\sigma \Big|_{x^0=0}, \tag{3.4}$$

$$0 = \partial_0 \left(\partial_\mu \gamma^\mu{}_\nu - \frac{1}{2} \partial_\nu \gamma_\sigma{}^\sigma \right) \Big|_{x^0=0}. \tag{3.5}$$

[2] In the sense that the components of ${}^{(s)}g$, and an appropriate number of their derivatives, with respect to any coordinate chart are smooth functions of s. The fact that such families of solutions exist is not a priori obvious, but happens to be true.

Proof. Let $F_\nu = \partial_\mu \gamma^\mu{}_\nu - \frac{1}{2}\partial_\nu \gamma_\sigma{}^\sigma$. Equation (3.3) is equivalent to $F_\nu = 0$. Notice that F_ν solves the wave equation for each ν:

$$\begin{aligned}\Box_\eta F_\nu &= \partial_\tau \partial^\tau \left(\partial_\mu \gamma^\mu{}_\nu - \frac{1}{2}\partial_\nu \gamma_\sigma{}^\sigma \right) \\ &= \partial_\mu \left(\Box_\eta \gamma^\mu{}_\nu \right) - \frac{1}{2}\partial_\nu \left(\Box_\eta \gamma_\sigma{}^\sigma \right) \\ &= 0.\end{aligned}$$

Now, by the uniqueness results of Chapter 1, we know that F_ν is uniquely determined by the value of F_ν, $\partial_0 F_\nu$ on the hyperplane $\{x^0 = 0\}$. In particular if $F_\nu = \partial_0 F_\nu = 0$ on $\{x^0 = 0\}$, then $F_\nu = 0$ in $\mathbb{R}^4$. □

What this result tells us is that it's enough to make sure that the constraint equations are satisfied at the initial time $x^0 = 0$, and the evolution equation (3.2) then ensures that the constraints *propagate* in time. The conditions (3.4), (3.5) will not hold for arbitrary choices of initial conditions $\gamma_{\mu\nu}|_{x^0=0}$ and $\partial_0 \gamma_{\mu\nu}|_{x^0=0}$, we need to restrict our choice of data to ensure that the initial constraints are satisfied.

Let us suppose that we are given $\phi, \beta_i, h_{ij}, k_{ij}$, for $i, j = 1, 2, 3$, which we assume to be smooth functions on $\mathbb{R}^3$. We suppose that the initial data for (3.2) is constructed from these functions in the following fashion:

$$\begin{aligned}\gamma_{00}|_{x^0=0} &= \phi \\ \gamma_{0i}|_{x^0=0} = \gamma_{i0}|_{x^0=0} &= \beta_i \\ \gamma_{ij}|_{x^0=0} &= h_{ij} \\ \partial_0 \gamma_{00}|_{x^0=0} &= 2k_{ii} \\ \partial_0 \gamma_{0j}|_{x^0=0} &= \partial_i h_{ij} - \frac{1}{2}\partial_j h_{ii} + \frac{1}{2}\partial_j \phi \\ \partial_0 \gamma_{ij}|_{x^0=0} &= -2k_{ij} + 2\partial_{(i}\beta_{j)}\end{aligned} \tag{3.6}$$

Lemma 3.2. *The solution $\gamma_{\mu\nu}$ to (3.2) with initial conditions (3.6) satisfies (3.3) throughout $\mathbb{R}^4$, and hence is a solution of the linearised Einstein equations, if and only if the following constraints hold on h_{ij}, k_{ij}:*

$$0 = \partial_i \partial_j h_{ij} - \partial_i \partial_i h_{jj}, \tag{3.7}$$
$$0 = \partial_i k_{ij} - \partial_j k_{ii}. \tag{3.8}$$

Proof. We first verify that constraint equation (3.4) is satisfied by our choice of initial data. Splitting into the time and space components, we first calculate:

$$\begin{aligned}\partial_\mu \gamma^\mu{}_0 - \frac{1}{2}\partial_0 \gamma_\sigma{}^\sigma \Big|_{x^0=0} &= -\partial_0 \gamma_{00} + \partial_i \gamma_{i0} + \frac{1}{2}\partial_0 \gamma_{00} - \frac{1}{2}\partial_0 \gamma_{ii} \Big|_{x^0=0} \\ &= -2k_{ii} + \partial_i \beta_i + k_{ii} - \frac{1}{2}\left(-2k_{ii} + 2\partial_i \beta_i\right) \\ &= 0.\end{aligned}$$

For the spacelike components we have:

$$\begin{aligned}
\partial_\mu \gamma^\mu{}_j - \frac{1}{2}\partial_j \gamma_\sigma{}^\sigma \Big|_{x^0=0} &= -\partial_0\gamma_{0j} + \partial_i\gamma_{ij} + \frac{1}{2}\partial_j\gamma_{00} - \frac{1}{2}\partial_j\gamma_{ii}\Big|_{x^0=0} \\
&= -\left.\partial_0\gamma_{0j}\right|_{x^0=0} + \partial_i h_{ij} + \frac{1}{2}\partial_j\phi - \frac{1}{2}\partial_j h_{ii} \\
&= 0.
\end{aligned}$$

Now we have to verify that (3.5) holds, at which point we are done by the previous Lemma. When we differentiate the constraint in the time direction, we will observe some components with two x^0-derivatives acting on them. To handle these, we use the fact that $\Box_\eta \gamma_{\mu\nu} = 0$, so that in particular:

$$\left.\partial_0\partial_0\gamma_{\mu\nu}\right|_{x^0=0} = \left.\partial_j\partial_j\gamma_{\mu\nu}\right|_{x^0=0}$$

We find that the $0-$component of (3.5) gives:

$$\begin{aligned}
\partial_0\left(\partial_\mu\gamma^\mu{}_0 - \frac{1}{2}\partial_0\gamma_\sigma{}^\sigma\right)\Big|_{x^0=0} &= -\partial_0\partial_0\gamma_{00} + \partial_0\partial_i\gamma_{i0} + \frac{1}{2}\partial_0\partial_0\gamma_{00} - \frac{1}{2}\partial_0\partial_0\gamma_{ii}\Big|_{x^0=0} \\
&= -\frac{1}{2}\partial_i\partial_i\phi + \partial_j\left(\partial_i h_{ij} - \frac{1}{2}\partial_j h_{ii} + \frac{1}{2}\partial_j\phi\right) - \frac{1}{2}\partial_j\partial_j h_{ii} \\
&= \partial_i\partial_j h_{ij} - \partial_i\partial_i h_{jj} \\
&= 0.
\end{aligned}$$

Where we use (3.7) in the last line. Finally, to verify the spacelike components of (3.5), we calculate:

$$\begin{aligned}
\partial_0\left(\partial_\mu\gamma^\mu{}_j - \frac{1}{2}\partial_j\gamma_\sigma{}^\sigma\right)\Big|_{x^0=0} &= -\partial_0\partial_0\gamma_{0j} + \partial_0\partial_i\gamma_{ij} + \frac{1}{2}\partial_j\partial_0\gamma_{00} - \frac{1}{2}\partial_j\partial_0\gamma_{ii}\Big|_{x^0=0} \\
&= -\partial_i\partial_i\beta_j + \partial_i\left(-2k_{ij} + \partial_i\beta_j + \partial_j\beta_i\right) \\
&\qquad + \partial_j k_{ii} - \frac{1}{2}\partial_j\left(-2k_{ii} + 2\partial_i\beta_i\right) \\
&= -2\left(\partial_i k_{ij} - \partial_j k_{ii}\right) \\
&= 0.
\end{aligned}$$

□

We can thus break the initial data down into geometrical objects defined on $\mathbb{R}^3$. We have two symmetric tensors, h and k, which have to obey the constraint equations (3.7), (3.8). We also have a scalar ϕ and a vector field β which are freely specifiable. Once we have specified these objects, there exists a unique solution γ to the equations (3.2) (3.3). We shall see later that h, k are *intrinsic* to the initial hypersurface $\{x^0 = 0\}$, while ϕ, β essentially encode information about the choice of coordinates (the wave coordinate condition doesn't fix completely fix the coordinates).

3.3 Hypersurface geometry and the constraint equations

[Before reading this section, you should make sure that you're familiar with the material in §A.3.4]

A note of caution We will specialise in this section to embedded spacelike submanifolds of Lorentzian manifolds. Many of the results have analogues in the case of embedded submanifolds of Riemannian manifolds. However, there are a few differences in sign introduced by the signature, so one should not assume that the formulae given here are directly valid in that situation.

3.3.1 The induced metric and second fundamental form

Let us suppose that we have a smooth, time oriented, four dimensional Lorentzian manifold $(\mathcal{M}, g)$. Suppose that we are also given a three dimensional manifold Σ. An *embedding* of Σ into $\mathcal{M}$ is a smooth map $\imath \in C^\infty(\Sigma; \mathcal{M})$, such that $\imath$ is a homeomorphism of Σ onto $\imath(\Sigma)$ and $\imath$ is an immersion, i.e. the push forward map $\imath_*$ acting on vectors is everywhere injective. As a result of the injectivity of $\imath_*$, we can identify $T_p\Sigma$ with a three-dimensional subspace $T_{\imath(p)}\imath(\Sigma) \subset T_{\imath(p)}\mathcal{M}$. We say that a vector $X \in T_{\imath(p)}\imath(\Sigma)$ is *tangent to* $\imath(\Sigma)$ *at* $\imath(p)$.

We assume now that an embedding has been fixed.

Definition 17. The metric induced on Σ by g is the pull-back of g to Σ by the embedding map $\imath$, and we denote the induced metric by $h := \imath^* g$. More concretely, for $X, Y \in T_p\Sigma$, we define:

$$h(X, Y) = g\left(\imath_* X, \imath_* Y\right)$$

We can translate our definitions of timelike/spacelike/null surfaces to the following:

Lemma 3.3. *The surface $\imath(\Sigma)$ is:*

i) Timelike *at $\imath(p)$ if and only if h is a Lorentzian metric at p.*

ii) Null *at $\imath(p)$ if and only if h is a degenerate quadratic form at p.*

iii) Spacelike *at $\imath(p)$ if and only if h is a Riemannian metric at p.*

We will mostly focus on the spacelike case, as this is the correct setting for an initial data surface for Einstein's equations. For each $p \in \Sigma$, there is a unique $N \in T_{\imath(p)}\mathcal{M}$ which is timelike, future directed, of unit length, and orthogonal to $T_{\imath(p)}\imath(\Sigma)$. Using the Canonical Immersion Theorem, Lemma A.8 in §A.3.4, we can assume that N is the restriction to $\imath(\Sigma)$ of a smooth vector field defined on $\mathcal{M}$. For any $V \in T_{\imath(p)}\mathcal{M}$, we define:

$$\top V := V + g(N, V)N, \qquad \perp V := -g(N, V)N$$

so that

$$V = \top V + \perp V$$

and we have $\top V \in T_{\imath(p)}\imath(\Sigma)$, and $\perp V$ is orthogonal to $T_{\imath(p)}\imath(\Sigma)$. In other words, $T_{\imath(p)}\mathcal{M}$ splits into

$$T_{\imath(p)}\mathcal{M} = T_{\imath(p)}\imath(\Sigma) \oplus N_{\imath(p)}\imath(\Sigma),$$

where $N_{\imath(p)}\imath(\Sigma) = \left(T_{\imath(p)}\imath(\Sigma)\right)^\perp$ is the orthogonal complement of $T_{\imath(p)}\imath(\Sigma)$ with respect to g. This is a one-dimensional timelike subspace, representing the normal directions to $\imath(\Sigma)$ with respect to the metric g.

We want to look at how the Levi-Civita connection ∇ behaves under this splitting. For this it will be useful to have the following result:

Lemma 3.4. *Let $\mathcal{U} \subset \Sigma$ be open, and suppose that $\imath : \Sigma \hookrightarrow \mathcal{M}$ is an embedding such that the image is spacelike.*

i) Suppose that $V_1, V_2, W \in \mathfrak{X}(\mathcal{M})$ satisfy $V_1 = V_2$ on $\imath(\mathcal{U})$. Then

$$\nabla_{V_1} W = \nabla_{V_2} W, \qquad \text{on} \imath(\mathcal{U}).$$

ii) Suppose that $W, V_1, V_2 \in \mathfrak{X}(\mathcal{M})$ satisfy $W \in T_{\imath(p)}\imath(\mathcal{U})$ for each $p \in \mathcal{U}$ and $V_1 = V_2$ on $\imath(\mathcal{U})$. Then

$$\nabla_W V_1 = \nabla_W V_2 \qquad \text{on} \imath(\mathcal{U}).$$

Proof. i) We have $V_1 - V_2 = 0$ on $\imath(\mathcal{U})$, so by the fact that ∇ is tensorial in its first slot, we have that on $\imath(\mathcal{U})$:

$$0 = \nabla_{V_1 - V_2} W = \nabla_{V_1} W - \nabla_{V_2} W$$

ii) It suffices to prove that if $V \in \mathfrak{X}(\mathcal{M})$ vanishes on $\imath(\mathcal{U})$, then $\nabla_W V = 0$ on $\imath(\mathcal{U})$. Let us fix some $K \in \mathfrak{X}(\mathcal{M})$ and define $f = g(V, K)|_{\imath(\mathcal{U})}$. Clearly $\imath^* f = 0$. Note also that there exists a vector field $X \in \mathfrak{X}(\mathcal{U})$ such that $\imath_* X = W$ on $\imath(\mathcal{U})$. We calculate that at $p \in \mathcal{U}$:

$$\begin{aligned} 0 &= X(\imath^* f)|_p \\ &= \imath_* X(f)|_{\imath(p)} = W(f)|_{\imath(p)} \\ &= W\left[g\left(V, K\right)\right]|_{\imath(p)} \\ &= g\left(\nabla_W V, K\right)|_{\imath(p)} + g\left(V, \nabla_W K\right)|_{\imath(p)} \\ &= g\left(\nabla_W V, K\right)|_{\imath(p)} \end{aligned}$$

Now, since K was arbitrary, we deduce $\nabla_W V|_{\imath(p)} = 0$. □

Suppose we have vector fields $X, Y \in \mathfrak{X}(\Sigma)$. By Corollary A.4, about any $p \in \Sigma$ we can find a neighbourhood $\mathcal{U}$ and two vector fields $\widetilde{X}, \widetilde{Y} \in \mathfrak{X}(\mathcal{M})$ such that $\imath_* X = \widetilde{X}$ and $\imath_* Y = \widetilde{Y}$ on $\imath(\mathcal{U})$, i.e. such that $\widetilde{X}, \widetilde{Y}$ extend X, Y away from $\imath(\mathcal{U})$. The previous Lemma shows that $\nabla_{\widetilde{X}} \widetilde{Y}\Big|_{\imath(\mathcal{U})}$ is independent of the extension, and depends only on X, Y.

Now, we can uniquely decompose:

$$\nabla_{\widetilde{X}} \widetilde{Y} = \top \nabla_{\widetilde{X}} \widetilde{Y} + \perp \nabla_{\widetilde{X}} \widetilde{Y}$$

where $\top$ is the tangential component and $\perp$ the normal.

Theorem 3.1. *i) Let $D : \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ be defined by:*

$$\imath_*(D_X Y) = \top \nabla_{\widetilde{X}} \widetilde{Y}$$

for all $X, Y \in \mathfrak{X}(\Sigma)$, where $\widetilde{X}, \widetilde{Y}$ are any (local) extensions of X, Y. Then D is the Levi-Civita connection of the induced metric h. (Note that the formula above determines $D_X Y$ uniquely by the infectivity of $\imath_*$).

ii) Let $k : \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \to C^\infty(\Sigma;\mathbb{R})$ be defined by:

$$k(X,Y) := \imath^* \left[g\left(N, \nabla_{\widetilde{X}} \widetilde{Y} \right) \right],$$

where $\widetilde{X}, \widetilde{Y}$ are any (local) extensions of X, Y. We have that

$$k(X,Y) = k(Y,X)$$

and

$$k(fX,Y) = fk(X,Y) \qquad \forall \quad f \in C^\infty(\Sigma,\mathbb{R}).$$

Proof. i) We first verify that D is a connection. By the linearity of the orthogonal projection and the push-forward map, we have

$$\begin{aligned} \imath_* \left(D_{X_1+X_2} Y \right) &= \top \nabla_{\widetilde{X}_1 + \widetilde{X}_2} \widetilde{Y} \\ &= \top \left(\nabla_{\widetilde{X}_1} \widetilde{Y} + \nabla_{\widetilde{X}_1} \widetilde{Y} \right) \\ &= \top \nabla_{\widetilde{X}_1} \widetilde{Y} + \top \nabla_{\widetilde{X}_1} \widetilde{Y} \\ &= \imath_*(D_{X_1} Y) + \imath_*(D_{X_2} Y) \\ &= \imath_*(D_{X_1} Y + D_{X_2} Y) \end{aligned}$$

so by the injectivity of $\imath_*$, we have $D_{X_1+X_2} Y = D_{X_1} Y + D_{X_2} Y$. A similar calculation shows:

$$D_X(Y_1 + Y_2) = D_X Y_1 + D_X Y_2.$$

We have to check the rules for $D_{fX} Y$ and $D_X(fY)$ hold. We note that by Corollary A.3 any $f \in C^\infty(\Sigma;\mathbb{R})$ can locally be written as $f = \imath^* \widetilde{f}$ for some $\widetilde{f} \in C^\infty(\mathcal{M};\mathbb{R})$. We calculate:

$$\begin{aligned} \imath_*(D_{fX} Y) &= \top \nabla_{\widetilde{f}\widetilde{X}} \widetilde{Y} \\ &= \top \left(\widetilde{f} \nabla_{\widetilde{X}} \widetilde{Y} \right) \\ &= \widetilde{f} \top \nabla_{\widetilde{X}} \widetilde{Y} = \imath^* \left(f D_X Y \right) \end{aligned}$$

similarly

$$\begin{aligned} \imath_*(D_X[fY]) &= \top \nabla_{\widetilde{X}} \widetilde{f}\widetilde{Y} \\ &= \top \left(\widetilde{f} \nabla_{\widetilde{X}} \widetilde{Y} + \widetilde{X}(\widetilde{f}) \widetilde{Y} \right) \\ &= \top \left(\widetilde{f} \nabla_{\widetilde{X}} \widetilde{Y} \right) + \widetilde{X}(\widetilde{f}) Y^* \\ &= \imath^* \left(f D_X Y + X(f) Y \right) \end{aligned}$$

Hence D is an affine connection. It remains to show that it is torsion free and metric. To verify that D is torsion free, we calculate

$$\begin{aligned} \imath_* \left(D_X Y - D_Y X - [X,Y] \right) &= \top \left(\nabla_{\widetilde{X}} \widetilde{Y} - \nabla_{\widetilde{Y}} \widetilde{X} \right) - [X,Y]^* \\ &= \top \left(\nabla_{\widetilde{X}} \widetilde{Y} - \nabla_{\widetilde{Y}} \widetilde{X} \right) - \left[\widetilde{X}, \widetilde{Y} \right] \\ &= \top \left(\nabla_{\widetilde{X}} \widetilde{Y} - \nabla_{\widetilde{Y}} \widetilde{X} - \left[\widetilde{X}, \widetilde{Y} \right] \right) = 0. \end{aligned}$$

Here we have used Lemma A.9. Finally, to check that D respects the induced metric h, we calculate:

$$\begin{aligned} X[h(Y,Z)] &= X[\imath^*(g(\widetilde{Y},\widetilde{Z}))] = \imath^*[\widetilde{X}g(\widetilde{Y},\widetilde{Z})] \\ &= \imath^*\left[g(\nabla_{\widetilde{X}}\widetilde{Y},\widetilde{Z}) + g(\widetilde{Y},\nabla_{\widetilde{X}}\widetilde{Z})\right] \\ &= \imath^*\left[g(\top\nabla_{\widetilde{X}}\widetilde{Y},\widetilde{Z}) + g(\widetilde{Y},\top\nabla_{\widetilde{X}}\widetilde{Z})\right] \\ &= \imath^*\left[g(\imath_*(D_XY),\widetilde{Z}) + g(\widetilde{Y},\imath_*(D_XZ))\right] \\ &= h\left(D_XY,Z\right) + h\left(Y,D_XZ\right). \end{aligned}$$

ii) We know by Lemma A.9 that $[\widetilde{X},\widetilde{Y}]$ is an extension of $[X,Y]$, so we have $[\widetilde{X},\widetilde{Y}]$ is tangent to $\imath(\Sigma)$, hence $g\left(N,[\widetilde{X},\widetilde{Y}]\right) = 0$. Thus:

$$0 = g\left(N,\nabla_{\widetilde{X}}\widetilde{Y} - \nabla_{\widetilde{Y}}\widetilde{X} - [\widetilde{X},\widetilde{Y}]\right) = g\left(N,\nabla_{\widetilde{X}}\widetilde{Y}\right) - g\left(N,\nabla_{\widetilde{Y}}\widetilde{X}\right)$$

so that

$$k(X,Y) = \imath^*\left[g\left(N,\nabla_{\widetilde{X}}\widetilde{Y}\right)\right] = \imath^*\left[g\left(N,\nabla_{\widetilde{Y}}\widetilde{X}\right)\right] = k(Y,X).$$

To establish the linearity, we calculate:

$$\begin{aligned} k(fX,Y) &= \imath^*\left[g\left(N,\nabla_{\widetilde{fX}}\widetilde{Y}\right)\right] \\ &= \imath^*\left[\widetilde{f}g\left(N,\nabla_{\widetilde{X}}\widetilde{Y}\right)\right] \\ &= \left(\imath^*\widetilde{f}\right)\imath^*\left[g\left(N,\nabla_{\widetilde{X}}\widetilde{Y}\right)\right] = fk(X,Y). \end{aligned}$$

Which establishes the result. □

From the second part of this theorem, we deduce that k is a $(0,2)-$tensor field defined on Σ, known as the *second fundamental form.* Notice that since $g(\widetilde{Y},N) = 0$ on $\imath(\Sigma)$, we must have that

$$\begin{aligned} 0 &= \widetilde{X}\left[g(\widetilde{Y},N)\right]\Big|_{\imath(\Sigma)} \\ &= \left[g\left(\nabla_{\widetilde{X}}N,\widetilde{Y}\right) + g\left(N,\nabla_{\widetilde{X}}\widetilde{Y}\right)\right]\Big|_{\imath(\Sigma)} \end{aligned}$$

So that we have:

$$k(X,Y) = -\imath^*\left[g\left(\nabla_{\widetilde{X}}N,\widetilde{Y}\right)\right], \tag{3.9}$$

which is known as *Weingarten's equation*, and gives an alternative approach to finding k.

Example 13. Suppose $\mathcal{M} = \mathbb{R}^4$ with coordinates $(t,x^i)_{i=1,2,3}$, and suppose that g is a Lorentzian metric on $\mathcal{M}$ given by:

$$g = -\phi(t,x)^2dt^2 + h_{ij}(t,x)dx^idx^j,$$

where $\phi > 0$ and h_{ij} is symmetric and positive definite for all (t, x). Consider $\Sigma = \mathbb{R}^3$ with coordinates $(y^i)_{i=1,2,3}$ and consider the map

$$\begin{array}{cccc} \imath : & \Sigma & \hookrightarrow & \mathcal{M} \\ & (y^i) & \mapsto & (0, y^i). \end{array}$$

So that $\imath(\Sigma) = \{t = 0\}$. Now, we note that if $f \in C^\infty(\mathcal{M}; \mathbb{R})$, then $\imath^* f(y) = f(0, y^i)$, so that pulling back a function from $\mathcal{M}$ to Σ simply consists of restricting f to $\{t = 0\}$. By considering the coordinate curves, we can also see that on $\{t = 0\}$ we have:

$$\imath_* \frac{\partial}{\partial y^i} = \frac{\partial}{\partial x^i}.$$

This suggests that the vector fields $\frac{\partial}{\partial x^i}$ are a suitable extension of $\frac{\partial}{\partial y^i}$.

Considering the pull-back of g to Σ, we find:

$$h\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right)\bigg|_y = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\bigg|_{\imath(y)} = h_{ij}(0, y)$$

so that

$$h := \imath^* g = h_{ij}(0, y) dy^i dy^j.$$

To find the second fundamental form, we first note that the future directed unit normal is:

$$N = \frac{1}{\phi}\frac{\partial}{\partial t},$$

which again admits an obvious extension away from $\{t = 0\}$. We calculate the second fundamental form as:

$$\begin{aligned} k\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right)\bigg|_y &= g\left(N, \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}\right)\bigg|_{\imath(y)} \\ &= g\left(\frac{1}{\phi}\frac{\partial}{\partial t}, \Gamma^\mu{}_{ij}\frac{\partial}{\partial x^\mu}\right)\bigg|_{\imath(y)} \\ &= -\phi \Gamma^t{}_{ij}\big|_{\imath(y)} \\ &= -\frac{\phi}{2} g^{t\mu}\left(\frac{\partial g_{i\mu}}{\partial x^j} + \frac{\partial g_{j\mu}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^\mu}\right)\bigg|_{\imath(y)} \\ &= -\frac{1}{2\phi(0, y)} \frac{\partial h_{ij}}{\partial t}(0, y). \end{aligned}$$

Thus the second fundamental form represents the 'first time derivative' of the induced metric. We might expect that the induced metric and the second fundamental form would represent the correct 'Cauchy data' for Einstein's equations, and we shall indeed see that this is the case.

3.3.2 The Gauss and Codazzi-Mainardi equations

The induced metric and second fundamental form carry information about how the surface Σ is 'glued into' the Lorentzian manifold $(\mathcal{M}, g)$. If the manifold satisfies some equations (for example Einstein's equations) then we expect that this is reflected in the information induced by g on Σ by the embedding map $\imath$. We shall see in this section that certain components of the curvature of g at $\imath(\Sigma)$ can be written in terms of h and k. This in turn will imply that when we impose conditions on g, this will be reflected as conditions on h, k.

Theorem 3.2 (Gauss' Equation)**.** *Let $(\mathcal{M}, g)$ be a smooth, time oriented spacetime, and let Σ be a three-dimensional manifold. Suppose that $\imath : \Sigma \hookrightarrow \mathcal{M}$ is an embedding of Σ such that $\imath(\Sigma)$ is spacelike. Suppose $\mathcal{U} \subset \Sigma$ is open. Let $X, Y, Z, W \in \mathfrak{X}(\mathcal{U})$ have extensions $\widetilde{X}, \widetilde{Y}, \widetilde{Z}, \widetilde{W} \in \mathfrak{X}(\mathcal{M})$ away from $\imath(\mathcal{U})$. Then:*

$$\begin{aligned} \imath^* \left[g \left({}^{\nabla}R(\widetilde{X}, \widetilde{Y})\widetilde{Z}, \widetilde{W} \right) \right] &= h \left({}^{D}R(X, Y)Z, W \right) \\ &\quad - k(X, Z)k(Y, W) + k(X, W)k(Y, Z), \end{aligned} \tag{3.10}$$

holds in $\mathcal{U}$, where ${}^{\nabla}R$, ${}^{D}R$ are the curvature operators corresponding to ∇, D respectively.

Proof. We will use the splitting of the connection ∇ induced by the embedding, as described in Theorem 3.1.

1. First note that
$$\nabla_{\widetilde{Y}} \widetilde{Z} = \top \nabla_{\widetilde{Y}} \widetilde{Z} - g(\nabla_{\widetilde{Y}} \widetilde{Z}, N)N. \tag{3.11}$$
Replacing $\widetilde{Y}$ with $\left[\widetilde{X}, \widetilde{Y}\right]$, and taking the inner product with $\widetilde{W}$, we have that on $\imath(\mathcal{U})$:
$$g\left(\nabla_{[\widetilde{X},\widetilde{Y}]}\widetilde{Z}, \widetilde{W}\right) = g\left(\top\nabla_{[\widetilde{X},\widetilde{Y}]}\widetilde{Z}, \widetilde{W}\right) = g\left(\imath_* D_{[X,Y]}Z, \imath_* W\right)$$
so that
$$\imath^* \left[g \left(\nabla_{[\widetilde{X},\widetilde{Y}]}\widetilde{Z}, \widetilde{W} \right) \right] = h\left(D_{[X,Y]}Z, W\right) \tag{3.12}$$

2. Differentiating (3.11) in the $\widetilde{X}$ direction and acting with $\top$, we have:
$$\top\left(\nabla_{\widetilde{X}}\nabla_{\widetilde{Y}}\widetilde{Z}\right) = \top\left(\nabla_{\widetilde{X}}\top\nabla_{\widetilde{Y}}\widetilde{Z}\right) - g(\nabla_{\widetilde{Y}}\widetilde{Z}, N)\nabla_{\widetilde{X}}N + \widetilde{X}\left[g(\nabla_{\widetilde{Y}}\widetilde{Z}, N)\right]N$$
We note that on $\imath(\mathcal{U})$, the first term is equal to $\imath_* D_X D_Y Z$, by Theorem 3.1 *i*), and the last term is in the normal direction. Now, let us take the inner product with $\widetilde{W}$ and pull-back by $\imath$ to obtain:
$$\begin{aligned} \imath^*\left[g\left(\nabla_{\widetilde{X}}\nabla_{\widetilde{Y}}\widetilde{Z}, \widetilde{W}\right)\right] &= \imath^*\left[g\left(\top\left(\nabla_{\widetilde{X}}\nabla_{\widetilde{Y}}\widetilde{Z}\right), \widetilde{W}\right)\right] \\ &= \imath^*\left[g\left(\top\left(\nabla_{\widetilde{X}}\top\nabla_{\widetilde{Y}}\widetilde{Z}\right), \widetilde{W}\right)\right] - \imath^*\left[g(\nabla_{\widetilde{Y}}\widetilde{Z}, N)g\left(N, \nabla_{\widetilde{X}}, \widetilde{W}\right)\right] \\ &= h(D_X D_Y Z, W) + k(Y, Z)k(X, W) \end{aligned} \tag{3.13}$$
Here, we have used the definition of k from Theorem 3.1 *i*), together with Weingarten's equation (3.9) to deal with the second term on the right hand side.

3. Now, we simply use the definition of ${}^{\nabla}R$:
$$ {}^{\nabla}R(\widetilde{X},\widetilde{Y})\widetilde{Z} = \nabla_{\widetilde{X}}\nabla_{\widetilde{Y}}\widetilde{Z} - \nabla_{\widetilde{Y}}\nabla_{\widetilde{X}}\widetilde{Z} - \nabla_{[\widetilde{X},\widetilde{Y}]}\widetilde{Z}. $$
Taking the inner product of this equation with $\widetilde{W}$, and pulling back by $\imath$, we deduce:
$$ \begin{aligned} \imath^*\left[g\left({}^{\nabla}R(\widetilde{X},\widetilde{Y})\widetilde{Z},\widetilde{W}\right)\right] &= \imath^*\left[g\left(\nabla_{\widetilde{X}}\nabla_{\widetilde{Y}}\widetilde{Z},\widetilde{W}\right)\right] - \imath^*\left[g\left(\nabla_{\widetilde{Y}}\nabla_{\widetilde{X}}\widetilde{Z},\widetilde{W}\right)\right] \\ &\quad - \imath^*\left[g\left(\nabla_{[\widetilde{X},\widetilde{Y}]}\widetilde{Z},\widetilde{W}\right)\right] \\ &= h(D_X D_Y Z, W) + k(Y,Z)k(X,W) \\ &\quad - h(D_Y D_Y X, W) - k(X,Z)k(Y,W) \\ &\quad - h\left(D_{[X,Y]}Z, W\right) \\ &= h\left({}^{D}R(X,Y)Z, W\right) \\ &\quad - k(X,Z)k(Y,W) + k(X,W)k(Y,Z), \end{aligned} $$
where we have used (3.12), (3.13) to pass from the first equality to the second equality. This is the result we require. □

Notice that the right hand side of Gauss' equation involves only geometric objects defined on the surface Σ. The left hand side is a quantity defined on the full spacetime. Note also that while all of the components of the Riemann tensor of h can appear on the right hand side, on the left hand side we can only realise purely tangential components of the Riemann tensor of g.

Gauss' equation tells us that the curvature of $(\mathcal{M}, g)$ in tangential directions to $\imath(\Sigma)$ is reflected both in the intrinsic curvature of Σ, thought of as a Riemannian manifold with metric h, as well as in the second fundamental form, which is sometimes referred to as the extrinsic curvature. This difference between intrinsic and extrinsic curvature is an important distinction even in the study of surfaces in $\mathbb{R}^3$. For example, a cylinder in $\mathbb{R}^3$ has no intrinsic curvature (the induced metric is flat) but it does have extrinsic curvature.

We shall also require another result relating the curvature of g to the intrinsic data h, k on Σ. This result involves components of the Riemann tensor of g which are not entirely tangential, but involve contraction with a normal direction.

Theorem 3.3 (Codazzi-Mainardi equation). *Let $(\mathcal{M}, g)$ be a smooth, time oriented spacetime, and let Σ be a three-dimensional manifold. Suppose that $\imath : \Sigma \hookrightarrow \mathcal{M}$ is an embedding of Σ such that $\imath(\Sigma)$ is spacelike and take $\mathcal{U} \subset \Sigma$ an open subset. Let $X, Y, Z \in \mathfrak{X}(\mathcal{U})$ have extensions $\widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \mathfrak{X}(\mathcal{M})$ away from $\imath(\mathcal{U})$ and suppose that $N \in \mathfrak{X}(\mathcal{M})$ agrees with the future directed unit normal on $\imath(\mathcal{U})$. Then:*
$$ \imath^*\left[g\left({}^{\nabla}R(\widetilde{X},\widetilde{Y})\widetilde{Z}, N\right)\right] = [D_X k](Y,Z) - [D_Y k](X,Z), \tag{3.14} $$
holds in $\mathcal{U}$

Proof. 1. Recall (3.11) from the proof of Gauss' equation
$$ \nabla_{\widetilde{Y}}\widetilde{Z} = \top\nabla_{\widetilde{Y}}\widetilde{Z} - g(\nabla_{\widetilde{Y}}\widetilde{Z}, N)N. \tag{3.15} $$

Replacing $\widetilde{Y}$ with $\left[\widetilde{X},\widetilde{Y}\right]$, and taking the inner product with N, we have that on $\imath(\mathcal{U})$:

$$g\left(\nabla_{[\widetilde{X},\widetilde{Y}]}\widetilde{Z},N\right)=g(\nabla_{[\widetilde{X},\widetilde{Y}]}\widetilde{Z},N)$$

so that

$$\imath^*\left[g\left(\nabla_{[\widetilde{X},\widetilde{Y}]}\widetilde{Z},N\right)\right]=k\left([X,Y],Z\right)=k(D_XY-D_YX,Z) \tag{3.16}$$

where we use the definition of k from Theorem 3.1 and the fact that D is torsion-free.

2. Differentiating (3.15) in the $\widetilde{X}$ direction and forming the inner product with N, we have:

$$\begin{aligned}g\left(\nabla_{\widetilde{X}}\nabla_{\widetilde{Y}}\widetilde{Z},N\right)=g\left(\nabla_{\widetilde{X}}{}^{\top}\nabla_{\widetilde{Y}}\widetilde{Z},N\right)-g\left(\nabla_{\widetilde{Y}}\widetilde{Z},N\right)g\left(\nabla_{\widetilde{X}}N,N\right)\\ +\widetilde{X}\left[g\left(\nabla_{\widetilde{Y}}\widetilde{Z},N\right)\right]\end{aligned} \tag{3.17}$$

Now, note that since $g(N,N)=-1$ on $\imath(\Sigma)$, and $\tilde{X}$ is tangent to $\imath(\Sigma)$, we have

$$0=\widetilde{X}\left[g(N,N)\right]\Big|_{\imath(\Sigma)}=2g\left(\nabla_{\widetilde{X}}N,N\right)\big|_{\imath(\Sigma)},$$

so that the second term on the right of (3.17) vanishes on $\imath(\Sigma)$. Pulling (3.17) back by $\imath$, we thus have:

$$\imath^*\left[g\left(\nabla_{\widetilde{X}}\nabla_{\widetilde{Y}}\widetilde{Z},N\right)\right]=k\left(D_YZ,X\right)+X\left[k(Y,Z)\right] \tag{3.18}$$

3. Now, we use the definition of ${}^{\nabla}R$:

$${}^{\nabla}R(\widetilde{X},\widetilde{Y})\widetilde{Z}=\nabla_{\widetilde{X}}\nabla_{\widetilde{Y}}\widetilde{Z}-\nabla_{\widetilde{Y}}\nabla_{\widetilde{X}}\widetilde{Z}-\nabla_{[\widetilde{X},\widetilde{Y}]}\widetilde{Z}.$$

Taking the inner product of this equation with N, and pulling back by $\imath$, we deduce:

$$\begin{aligned}\imath^*\left[g\left({}^{\nabla}R(\widetilde{X},\widetilde{Y})\widetilde{Z},N\right)\right]&=\imath^*\left[g\left(\nabla_{\widetilde{X}}\nabla_{\widetilde{Y}}\widetilde{Z},N\right)\right]-\imath^*\left[g\left(\nabla_{\widetilde{Y}}\nabla_{\widetilde{X}}\widetilde{Z},N\right)\right]\\&\quad-\imath^*\left[g\left(\nabla_{[\widetilde{X},\widetilde{Y}]}\widetilde{Z},N\right)\right]\\&=k\left(D_YZ,X\right)+X\left[k(Y,Z)\right]\\&\quad-k\left(D_XZ,Y\right)-Y\left[k(Y,Z)\right]\\&\quad-k(D_XY-D_YX,Z)\\&=X\left[k(Y,Z)\right]-k\left(D_XZ,Y\right)-k(Z,D_XY)\\&\quad-\left(Y\left[k(Y,Z)\right]-k\left(D_YZ,X\right)-k(Z,D_YX)\right)\\&=\left[D_Xk\right](Y,Z)-\left[D_Yk\right](X,Z).\end{aligned}$$

Here we have used (3.16), (3.18) to pass from the first inequality to the second, and the definition of ∇_Xk, ∇_Yk for the final equality. This is the expression we require. □

The calculations in this section are somewhat involved, but the basic idea is to use and reuse the splitting of the connection into tangential and normal parts that we discussed in Theorem 3.1. The main things to take away are the equations (3.10), (3.14) which allow us to relate certain components of the curvature of g to the objects induced on Σ by $\imath$, namely h and k.

3.3.3 The Einstein constraint equations

Now, since we have expressed certain components of the Riemann tensor of g in terms of the quantities h, k, it is natural that imposing conditions on the Riemann tensor of g will impose conditions on h and k. In particular, if we assume that the metric g satisfies Einstein's equations, we shall see that certain relations must hold between h and k. These are the *Einstein constraint equations.*

Let us suppose $\mathcal{U} \subset \Sigma$ is open and that $\{e_i\}_{i=1,2,3}$, where $e_i \in \mathfrak{X}(\mathcal{U})$, form a local basis which is orthonormal with respect to h. We can assume that there exist $\widetilde{e}_i \in \mathfrak{X}(\mathcal{M})$ which extend e_i away from $\imath(\mathcal{U})$. Setting $\widetilde{e}_0 = N$, we have that $\{\widetilde{e}_\mu\}_{\mu=0,\ldots,3}$ is a basis on some neighbourhood of $\imath(\mathcal{U}) \subset \mathcal{M}$, which is orthonormal at each point of $\imath(\mathcal{U})$.

Let us now take traces of the Gauss equation (3.10) to relate the Ricci curvature of g to quantities defined on Σ. We have:

$$\begin{aligned}
\imath^*\left[g\left({}^\nabla R(\widetilde{e}_i,\widetilde{Y})\widetilde{Z},\widetilde{e}_j\right)\delta^{ij}\right] &= \imath^*\left[g\left({}^\nabla R(\widetilde{e}_\mu,\widetilde{Y})\widetilde{Z},\widetilde{e}_\nu\right)\eta^{\mu\nu}\right] + \imath^*\left[g\left({}^\nabla R(\widetilde{e}_0,\widetilde{Y})\widetilde{Z},\widetilde{e}_0\right)\right]\\
&= \imath^*\left[Ric_g(\widetilde{Y},\widetilde{Z})\right] + \imath^*\left[g\left({}^\nabla R(\widetilde{e}_0,\widetilde{Y})\widetilde{Z},\widetilde{e}_0\right)\right].
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\imath^*\left[g\left({}^\nabla R(\widetilde{e}_i,\widetilde{Y})\widetilde{Z},\widetilde{e}_j\right)\delta^{ij}\right] &= \delta^{ij}h\left({}^D R(e_i,Y)Z,e_j\right)\\
&\qquad - \delta^{ij}k(e_i,Z)k(e_j,Y) + \delta^{ij}k(e_i,e_j)k(Y,Z)\\
&= Ric_h(Y,Z) - \delta^{ij}k(e_i,Z)k(e_j,Y) + (\mathrm{Tr}_h k)\,k(Y,Z)
\end{aligned}$$

So that:

$$\begin{aligned}
&\imath^*\left[Ric_g(\widetilde{Y},\widetilde{Z})\right] + \imath^*\left[g\left({}^\nabla R(\widetilde{e}_0,\widetilde{Y})\widetilde{Z},\widetilde{e}_0\right)\right]\\
&\qquad = Ric_h(Y,Z) - \delta^{ij}k(e_i,Z)k(e_j,Y) + (\mathrm{Tr}_h k)\,k(Y,Z)
\end{aligned}$$

Einstein's equations don't impose a condition on the left hand side of this equation, so we trace again over the Y, Z slots. We have:

$$\begin{aligned}
&\imath^*\left[Ric_g(\widetilde{e}_k,\widetilde{e}_l)\right]\delta^{kl} + \imath^*\left[g\left({}^\nabla R(\widetilde{e}_0,\widetilde{e}_k)\widetilde{e}_l,\widetilde{e}_0\right)\delta^{kl}\right]\\
&\qquad = \imath^*\left[Ric_g(\widetilde{e}_k,\widetilde{e}_l)\right]\delta^{kl} + \imath^*\left[g\left({}^\nabla R(\widetilde{e}_0,\widetilde{e}_\sigma)\widetilde{e}_\tau,\widetilde{e}_0\right)\eta^{\sigma\tau}\right] + \imath^*\left[g\left({}^\nabla R(\widetilde{e}_0,\widetilde{e}_0)\widetilde{e}_0,\widetilde{e}_0\right)\right]\\
&\qquad = \imath^*\left[Ric_g(\widetilde{e}_k,\widetilde{e}_l)\right]\delta^{kl} + \imath^*\left[Ric_g(\widetilde{e}_0,\widetilde{e}_0)\right]\\
&\qquad = \imath^*\left[R_g + 2Ric_g(\widetilde{e}_0,\widetilde{e}_0)\right],
\end{aligned}$$

where we have made use of the symmetries of the Riemann tensor in several places. We also have:

$$\begin{aligned} &Ric_h(e_k, e_k)\delta^{kl} - \delta^{ij}\delta^{kl}k(e_i, e_k)k(e_j, e_l) + (\mathrm{Tr}_h k)\, k(e_k, e_l)\delta^{kl} \\ &\qquad = R_h - |k|_h^2 + (\mathrm{Tr}_h k)^2 \end{aligned}$$

Let us suppose that g satisfies Einstein's equations, with some energy-momentum tensor T. Notice that on $\imath(\Sigma)$:

$$R_g + 2Ric_g(\widetilde{e}_0, \widetilde{e}_0) = 2\left(Ric_g - \frac{1}{2}R_g g\right)(\tilde{e}_0, \tilde{e}_0) = 2T(\tilde{e}_0, \tilde{e}_0) + 2\Lambda.$$

Now, $T(\tilde{e}_0, \tilde{e}_0)$ is the energy density of the matter fields, as measured by an observer whose instantaneous velocity is given by $\widetilde{e}_0$. We introduce $\rho \in C^\infty(\Sigma; \mathbb{R})$, the local energy density of matter fields on Σ, defined by:

$$\rho := \imath^*\left[T(\tilde{e}_0, \tilde{e}_0)\right].$$

Then, putting everything together, we have the first Einstein constraint equation:

$$R_h - |k|_h^2 + (\mathrm{Tr}_h k)^2 = 2\rho + 2\Lambda \tag{3.19}$$

In particular, in the vacuum case with vanishing cosmological constant, we have:

$$R_h - |k|_h^2 + (\mathrm{Tr}_h k)^2 = 0.$$

We can perform a similar procedure with the Codazzi-Mainardi equation (3.14). We have:

$$\begin{aligned} \imath^*\left[g\left({}^\nabla R(\widetilde{X}, \widetilde{e}_i)\widetilde{e}_j, \widetilde{e}_0\right)\right]\delta^{ij} &= \imath^*\left[g\left({}^\nabla R(\widetilde{X}, \widetilde{e}_\mu)\widetilde{e}_\nu, \widetilde{e}_0\right)\eta^{\mu\nu}\right] + \imath^*\left[g\left({}^\nabla R(\widetilde{X}, \widetilde{e}_0)\widetilde{e}_0, \widetilde{e}_0\right)\right] \\ &= \imath^*\left[Ric_g\left(\widetilde{X}, \widetilde{e}_0\right)\right] \end{aligned}$$

We also have:

$$\left[D_X k\right](e_i, e_j)\delta^{ij} - \left[D_{e_i} k\right](X, e_j)\delta^{ij} = X\left[\mathrm{Tr}_h K\right] - \mathrm{div}_h k\,(X),$$

where we are using that D is the Levi-Civita connection of h and the fact that e_i is an orthonormal basis for h. Since $\widetilde{e}_0$ is normal to $\imath(\Sigma)$, we have that on $\imath(\Sigma)$:

$$Ric_g\left(\widetilde{X}, \widetilde{e}_0\right) = \left(Ric_g - \frac{1}{2}R_g g\right)(\widetilde{X}, \widetilde{e}_0) = T(\widetilde{X}, \widetilde{e}_0).$$

Now $-T(\cdot, \widetilde{e}_0)$, which we think of as a one-form, is the momentum flux density of the matter fields, as measured by an observer whose instantaneous velocity is given by $\widetilde{e}_0$. We introduce $J \in \mathfrak{X}^*(\Sigma)$, the local momentum flux density of matter fields on Σ, defined by:

$$J := -\imath^*\left[T(\cdot, \widetilde{e}_0)\right].$$

Putting this together with the Codazzi-Mainardi equation, we deduce the second Einstein constraint equation:

$$\mathrm{div}_h k - d\,(\mathrm{Tr}_h K) = J.$$

In particular, in the vacuum case with vanishing cosmological constant, we have:

$$\mathrm{div}_h k - d\,(\mathrm{Tr}_h K) = 0.$$

Exercise 3.7. Recall the Schwarzschild metric in Painlevé-Gullstrand coordinates (Examples 7, 9, 12) is a Lorentzian metric on $\mathcal{M} = \mathbb{R} \times (0,\infty) \times S^2$ given by:

$$g = -\left(1 - \frac{2m}{r}\right) dt^2 + 2\sqrt{\frac{2m}{r}} dtdr + dr^2 + r^2 g_{S^2},$$

And let us choose the local basis of vector fields $\{e_\mu\}$, as in Examples 7, 9. Let $\Sigma = (0,\infty) \times S^2 \simeq \mathbb{R}^3 \setminus \{0\}$. Consider the map

$$\begin{array}{rccc} \imath & : & \Sigma & \hookrightarrow & \mathcal{M} \\ & & (\rho, \boldsymbol{X}) & \mapsto & (0, \rho, \boldsymbol{X}). \end{array}$$

In other words, the suface $\imath(\Sigma)$ is the surface $\{t = 0\}$.

a) Show that the future directed unit normal of $\imath(\Sigma)$ is given by:

$$N = e_0 = \frac{\partial}{\partial t} - \sqrt{\frac{2m}{r}} \frac{\partial}{\partial r}.$$

b) Show that the induced metric h is the canonical flat metric on $\mathbb{R}^3 \setminus \{0\}$:

$$h = d\rho^2 + \rho^2 g_{S^2}.$$

c) Show that:

$$k = -\frac{1}{2\rho}\sqrt{\frac{2m}{\rho}} d\rho^2 + \rho\sqrt{\frac{2m}{\rho}} g_{S^2}.$$

[Hint: consider $k(b_i, b_j)$ for a suitable basis of vector fields on Σ such that $\imath^* b_i = e_i$, and use (2.13)]

d*) Under a change of coordinates $\boldsymbol{x} = \rho \boldsymbol{X}$ from polar to Cartesian coordinates, you are given that h and k become:

$$h = \delta_{ij} dx^i dx^j,$$

$$k = \sqrt{\frac{2m}{|\boldsymbol{x}|^3}} \left(\delta_{ij} - \frac{3}{2}\frac{x_i x_j}{|\boldsymbol{x}|^2}\right) dx^i dx^j.$$

Show that:

$$|k|_h^2 - (\mathrm{Tr}_h k)^2 = 0,$$

and

$$\mathrm{div}_h k - d\left(\mathrm{Tr}_h k\right) = 0.$$

[Hint: Note that since h is the canonical metric on $\mathbb{R}^3$ in Cartesian coordinates, $(\mathrm{div}_h k)_j = \partial_i k_{ij}$ and $(df)_i = \partial_i f$.]

3.4 The Cauchy problem

For the purposes of studying the vacuum Einstein equations, we can summarise the results of the previous section in the following theorem:

Theorem 3.4. *Let $(\mathcal{M}, g)$ be a smooth, time oriented spacetime satisfying the vacuum Einstein equations with vanishing cosmological constant:*

$$Ric_g = 0.$$

Suppose that Σ is a smooth three dimensional manifold and that $\imath : \Sigma \hookrightarrow \mathcal{M}$ is a smooth embedding whose image is everywhere spacelike. Then the metric h and second fundamental form k induced on Σ by $\imath$ satisfy the Einstein constraint equations:

$$0 = R_h - |k|_h^2 + (\mathrm{Tr}_h k)^2 , \tag{3.20}$$

$$0 = \mathrm{div}_h k - d\,(\mathrm{Tr}_h k) . \tag{3.21}$$

We want to consider the Cauchy problem for Einstein's equations. Loosely, we wish to specify data on some initial hypersurface, and then construct a solution which represents the evolution of that data into the future. Recall that in the case of a field ψ satisfying the wave equation, the correct Cauchy data on a spacelike hypersurface, Σ, was $\psi|_\Sigma$ and $N_\Sigma \psi|_\Sigma$. In the case of Einstein's equations, a natural candidate to take the place of $\psi|_\Sigma$ is h, the induced metric. By considering Example 13, we can see that a natural candidate to take the place of $N_\Sigma \psi|_\Sigma$ is k, the second fundamental form. Theorem 3.4 gives some necessary conditions on h and k such that they represent initial data for Einstein's equations. We shall see that these conditions are in fact sufficient.

Definition 18. An *admissible triple* (Σ, h, k) consists of a smooth $3-$dimensional manifold Σ, equipped with a Riemannian metric h and a symmetric $(0, 2)-$tensor k satisfying the Einstein constraint equations (3.20), (3.21).

Examples of admissible triples include[3] $(\mathbb{R}^3, \delta, 0)$, the data induced on the surface $\{t = 0\}$ in the Minkowski spacetime, as well as the example constructed in Exercise 3.7.

You should think of an admissible triple as giving the 'initial conditions' for Einstein's equations. In contrast to the case of the wave equation, there is a subtlety in defining what we mean by a solution with this initial data. This comes about because we don't know a priori the spacetime manifold $\mathcal{M}$ on which we shall solve Einstein's equations. The correct notion of solution is given by:

Definition 19. Suppose (Σ, h, k) is an admissible triple. A *development* of (Σ, h, k) is a Lorentzian manifold $(\mathcal{M}, g)$, together with an embedding map $\imath : \Sigma \hookrightarrow \mathcal{M}$ such that

i) g satisfies the vacuum Einstein equations in $\mathcal{M}$:

$$Ric_g = 0$$

[3]Here $\delta = \delta_{ij} dx^i dx^j$ is the flat metric on $\mathbb{R}^3$

ii) h is the metric induced by g on Σ under the embedding map $\imath$.

iii) k is the second fundamental form of the embedding $\imath$.

iv) $\mathcal{M}$ is the future Cauchy development of $\imath(\Sigma)$: i.e. $D^+(\Sigma) = \mathcal{M}$.

By 'solving' Einstein's equations with a certain admissible triple as initial data, we mean finding a development of the triple. Before we are able to state the well posedness of Einstein's equations, we need one more ingredient. In order to say that a PDE problem is well posed, we not only require that a solution exists for given initial data, but that moreover it is unique. In the case of Einstein's equations, this is somewhat subtle, because the 'solution' we construct is a geometrical object. We know that the same geometric object can be described in several different ways.

To motivate our next definition, consider $\mathcal{M} = (0, \infty) \times (-\pi, \pi)$ with coordinates (r, θ), endowed with the metric

$$g = dr^2 + r^2 d\theta^2.$$

On the other hand, consider $\mathcal{M}' = \mathbb{R}^2 \setminus \{x \leq 0, y = 0\}$ where (x, y) are the usual coordinates on $\mathbb{R}^2$. We endow $\mathcal{M}'$ with the flat Riemannian metric:

$$g' = dx^2 + dy^2.$$

There is a map between these two manifolds, given by:

$$\begin{array}{rccc} \jmath : & \mathcal{M} & \to & \mathcal{M}' \\ & (r, \theta) & \mapsto & (r \sin\theta, r\cos\theta) \end{array}$$

The map $\jmath$ is smooth, bijective and has smooth inverse, hence $\jmath$ is a diffeomorphism. Moreover, we can verify that

$$\jmath^* g' = g.$$

We say that $\jmath$ is an isometry. Although $(\mathcal{M}, g)$ and $(\mathcal{M}', g')$ are *different* manifolds, we nevertheless think of them as describing the *same* underlying geometry, but in different coordinates.

The solutions that we construct to Einstein's equations will be unique up to transformations of this kind, and extensions.

Definition 20. An isometric embedding from $(\mathcal{M}, g)$ to $(\mathcal{M}', g')$ is an embedding $\jmath : \mathcal{M} \hookrightarrow \mathcal{M}'$ such that $\jmath^* g' = g$.

We are now ready to state the main result of this course:

Theorem 3.5 (Choquet-Bruhat–Geroch). *Given an admissible triple (Σ, h, k), there exists a unique development $(\mathcal{M}, g, \imath)$ which is maximal in the sense that if $(\widetilde{\mathcal{M}}, \widetilde{g}, \widetilde{\imath})$ is any other development of (Σ, h, k), then there exists an isometric embedding $\jmath : \widetilde{\mathcal{M}} \hookrightarrow \mathcal{M}$ such that:*

$$\jmath \circ \tilde{\imath} = \imath.$$

The full proof of this theorem is beyond the scope of the course, but is contained in the book of Choquet-Bruhat[4]. We shall offer a sketch of the proof, which follows a similar structure to our discussion of the linearised problem around Minkowski space.

Sketch proof:

1. We assume there exist global coordinates[5] (x^i) on Σ and extend them to coordinates (t, x^i) on $\mathcal{M} := (-\epsilon, \epsilon) \times \Sigma$. Recall from Theorem 2.9 that Einstein's equations in $\mathcal{M}$ are equivalent to the system of quasilinear wave equations:
$$-\frac{1}{2}g^{\mu\alpha}\frac{\partial^2 g_{\sigma\nu}}{\partial x^\mu \partial x^\alpha} + P_{\sigma\nu}(g, \partial g) = 0 \tag{3.22}$$
provided that the wave coordinate condition holds:
$$\Gamma^\alpha{}_\mu{}^\mu = 0. \tag{3.23}$$

2. By some standard results in the study of nonlinear wave equations, there exists a unique solution to (3.22) with the initial conditions:
$$g|_{t=0} = -dt^2 + h \tag{3.24}$$
$$\partial_t g|_{t=0} = -2k \tag{3.25}$$
provided $\epsilon > 0$ is sufficiently small. Notice that $g|_{t=0}$ is Lorentzian by construction.

3. We define $F^\alpha := \Gamma^\alpha{}_\mu{}^\mu$, and then show that if (3.22) is satisfied, then F^α satisfies a (linear) system of wave equations:
$$\Box_g F^\alpha + (A \cdot F)^\alpha = 0.$$
By similar methods to those used in the proof of Proposition 1 we can show that if $F^\alpha|_{t=0} = \partial_t F^\alpha|_{t=0} = 0$, then $F^\alpha \equiv 0$.

4. We next demonstrate that the condition $F^\alpha|_{t=0} = \partial_t F^\alpha|_{t=0} = 0$, then $F^\alpha \equiv 0$ is equivalent to the constraint equations holding on h, k. By restricting to the future domain of dependence of $\{0\} \times \Sigma$ we have constructed a development of (Σ, h, k).

5. To establish local uniqueness we show that given *any* development of (Σ, h, k) it is possible to construct wave coordinates such that (3.24), (3.25) hold. By the uniqueness of solutions of (3.22), we deduce that given any two developments there is a neighbourhood of the initial surface in each which can mapped onto one another by an isometry.

6. The final stage is to establish the existence of a single *maximal* development. Historically, this was the final part of the result to be established. The issue here is that while we know that in a neighbourhood of Σ two developments are isometric, constructing a larger development in which both embed isometrically is difficult.

[4]"General Relativity and the Einstein Equations", Yvonne Choquet-Bruhat, Oxford 2009. See Chapter VI, §7, 8, 9

[5]This is not a significant restriction by the finite speed of propagation property for hyperbolic PDE.

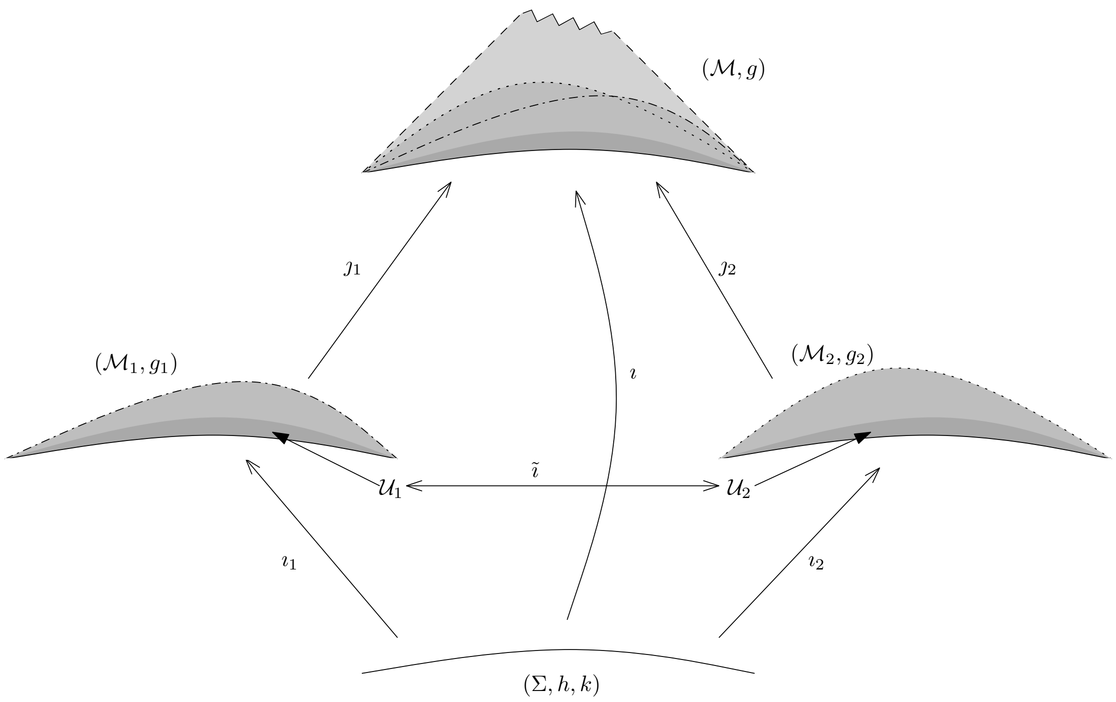

Figure 3.1 Two developments $(\mathcal{M}_i, g_i, \imath_i)$ of an admissible triple, and the maximal development $(\mathcal{M}, g, \imath)$.

The situation is shown in Figure 3.1. Here we have two developments $(\mathcal{M}_1, g_1, \imath_1)$, $(\mathcal{M}_2, g_2, \imath_2)$ of an admissible triple (Σ, h, k). By the previous part we can deduce that there is a neighbourhood $\mathcal{U}_1 \subset \mathcal{M}_1$ of $\imath_1(\Sigma)$, and a neighbourhood $\mathcal{U}_2 \subset \mathcal{M}_2$ of $\imath_2(\Sigma)$ which are mapped isometrically onto one another by $\tilde{\imath}$. The uniqueness theorem states that there exists a larger development, $(\mathcal{M}, g, \imath)$ such that $(\mathcal{M}_i, g_i)$ is isometrically embedded into $(\mathcal{M}, g)$ by the maps $\jmath_i$. Moreover, the map $\jmath_i \circ \imath_i$ which embeds Σ into $\mathcal{M}$ should be the same as $\imath$.

In the original paper of Geroch and Choquet-Bruhat[6] the construction of the maximal development was accomplished by an appeal to Zorn's Lemma (and hence the full Axiom of Choice). Recently the proof has been 'deZornified' by Sbierski[7]. □

While the theorem establishes the existence of a maximal development, it doesn't tell us anything about what the obstacles to extending the solution beyond that development are. For example, the maximal global development of $(\mathbb{R}^3, \delta, 0)$, is the whole of Minkowski space to the future of $\{t = 0\}$: a future complete manifold (i.e., any future directed timelike curve can be extended indefinitely). By contrast, the maximal development of the data constructed in Exercise (3.7) is the region of the Schwarzschild space-time in

[6]"Global aspects of the Cauchy problem in general relativity", Yvonne Choquet-Bruhat, Robert Geroch, Comm. Math. Phys. 14 (1969) p329

[7]"On the Existence of a Maximal Cauchy Development for the Einstein Equations - a Dezornification", Jan Sbierski, arXiv:1309.7591

Painlevé-Gullstrand coordinates to the future of $\{t = 0\}$. This is not future complete, as future directed timelike curves can meet the curvature singularity at $r = 0$.

Much of the current research activity in Mathematical Relativity concerns the global properties of solutions which start from initial data 'close' to the data of an explicitly known solution. For example, one of the crowning achievements of recent work is the following result:

Theorem 3.6 (Christodoulou–Klainerman, '92)**.** *Suppose that we start with an admissible triple* $(\mathbb{R}^3, h, k)$*, with*

$$||h - \delta||_a + ||k||_b < \epsilon,$$

for $\epsilon > 0$ *sufficiently small, where* $||\cdot||_a, ||\cdot||_b$ *are suitable norms*[8]*. Then the maximal development is future complete, and asymptotic to the Minkowski spacetime.*

This result is important because it asserts that for a sufficiently 'small' gravitational field, i.e. a field which is initially sufficiently close to flat space, no singularities form in the evolution. In particular, no black holes are formed.

By contrast, an analogous result is not known for the case of the Schwarzschild black hole discussed above. It is believed that data sufficiently close to the Schwarzschild data constructed in Exercise 3.7 will evolve to give a spacetime containing a region similar to the exterior region, $r > 2m$, of Schwarzschild, however this is at present an unproven conjecture.

[8] In fact, weighted Sobolev norms.