An algorithm for efficient detection of (N, N)-splittings and its application to the isogeny problem in dimension 2

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In this work we look at the problem in dimension 2 and decrease the concrete complexity of the best attack due to Costello–Smith.

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For the purposes of this talk, we only need to keep in mind that there are two types of surfaces: "reducible" and "non-reducible".

General Isogeny Problem in Two Dimensions

In its most general form, the superspecial isogeny problem in two dimensions asks to find an isogeny

$$\phi\colon A \longrightarrow A',$$

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The general isogeny problem can be viewed as finding a path between two nodes in the superspecial isogeny graph.

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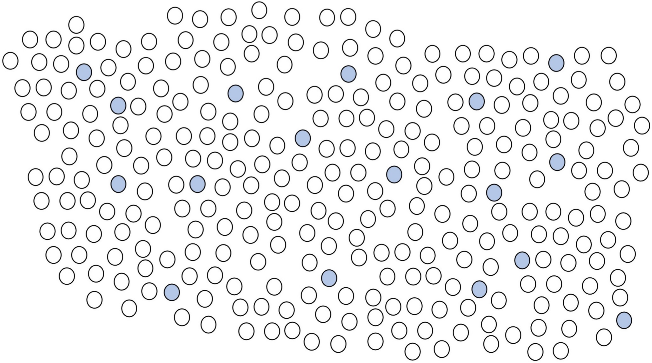
$\mathcal{S}(p)$ is equal to the disjoint union of:

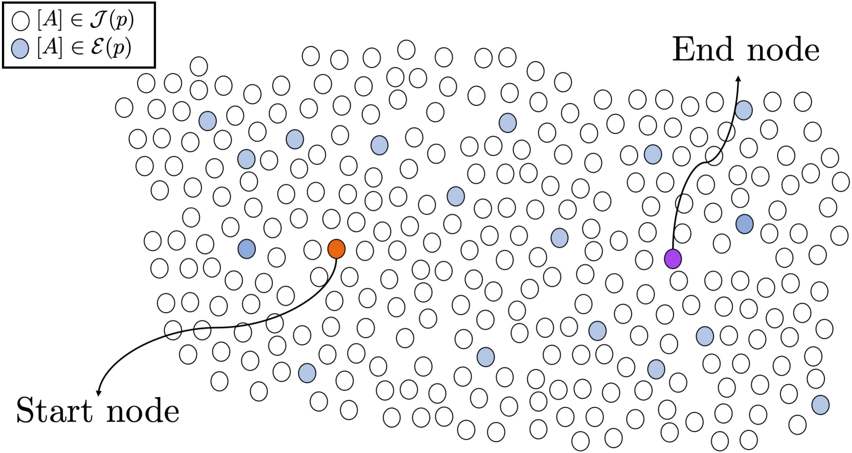
$$\begin{aligned}
\mathcal{E}(p) &:= \{A \in \mathcal{S}(p) \ : \ A \cong E \times E' \text{ with } E,\ E' \text{ supersingular ECs}\}. \\
\mathcal{J}(p) &:= \mathcal{S}(p) \setminus \mathcal{E}(p) \\
&= \{A \in \mathcal{S}(p) \ : \ A \cong \mathrm{Jac}(C)\}
\end{aligned}$$

The Superspecial Isogeny Graph $\Gamma(N; p)$

$O(p^3)$ nodes with ○ $A \in \mathcal{J}(p)$
$O(p^2)$ nodes with ● $A \in \mathcal{E}(p)$



Attacking the General Isogeny Problem: Costello–Smith



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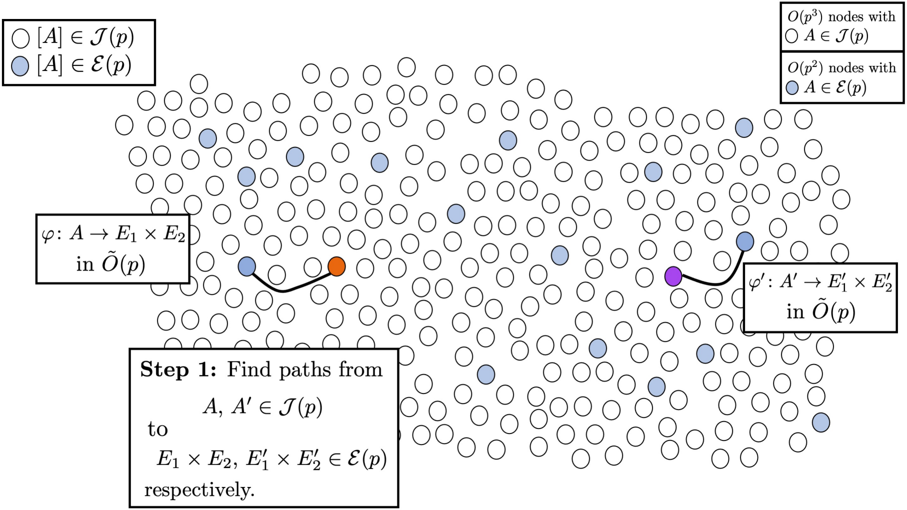

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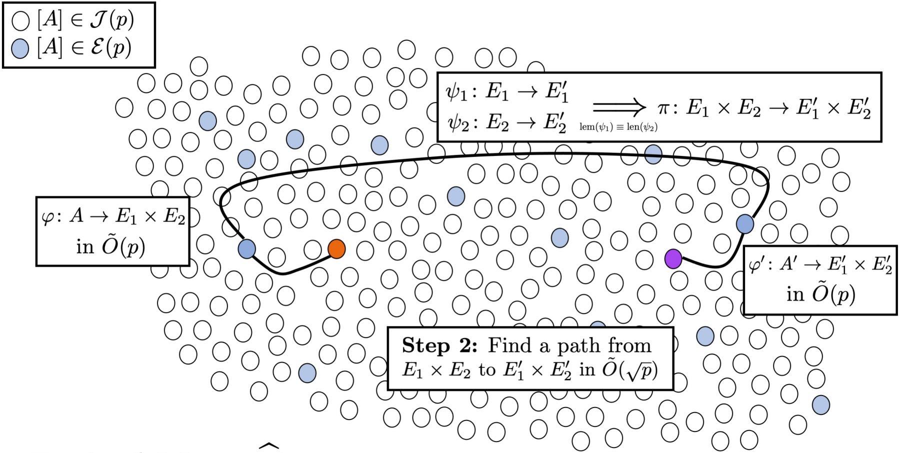

Desired Map: $\widehat{\varphi'} \circ \pi \circ \varphi$

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Question: Taking steps in $\Gamma(2;p)$, can we detect whether the current node A_i is in (N,N)-split (i.e., (N,N)-isogenous to a product) for $N > 2$?

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Naive Answer: Compute all (N,N)-isogenies from A_i, but this is not efficient. Can we make the detection efficient?

Detecting an (N, N)-splitting

There exist (easily computable) functions $\boldsymbol{\alpha}(A) = (\alpha_1(A), \alpha_2(A), \alpha_3(A))$ which assigns to A a triple of elements of $\mathbb{F}_{p^2}$ which uniquely determine $A^\dagger$.

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For $N \leq 11$, Kumar [Kum15] provides rational functions $i_1(r, s)$, $i_2(r, s)$, $i_3(r, s) \in \mathbb{F}_p(r, s)$, such that if there exists a simultaneous solution $r_0, s_0 \in \bar{\mathbb{F}}_p$ of

$$\begin{cases} i_1(r, s) = \alpha_1(A) \\ i_2(r, s) = \alpha_2(A) \\ i_3(r, s) = \alpha_3(A) \end{cases}$$

and the denominators do not vanish at (r_0, s_0), then A is (N, N)-split.

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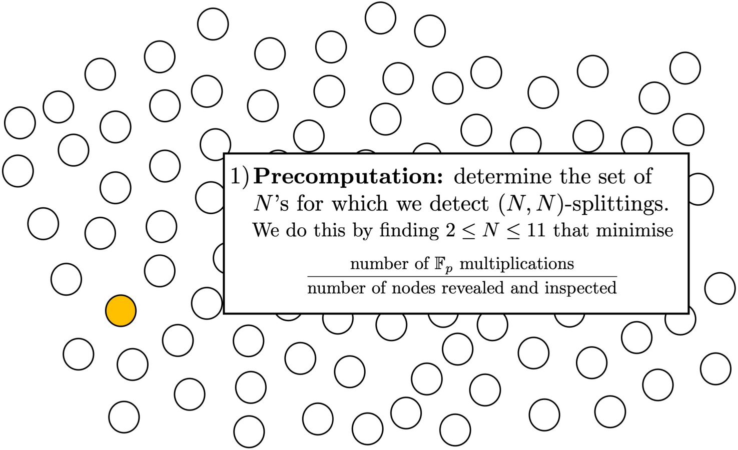
In fact, we obtain a more efficient method by precomputing the resultants generically.

Attacking the General Isogeny Problem: Revisted

We now apply efficient splitting detection to the Costello–Smith algorithm and decreasing its concrete complexity.



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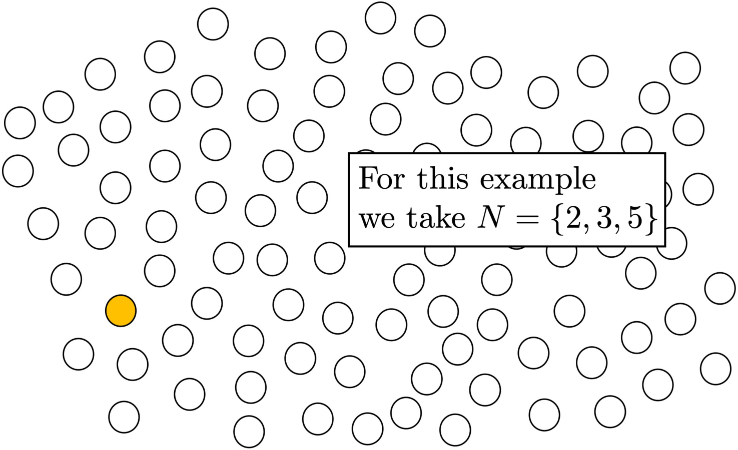

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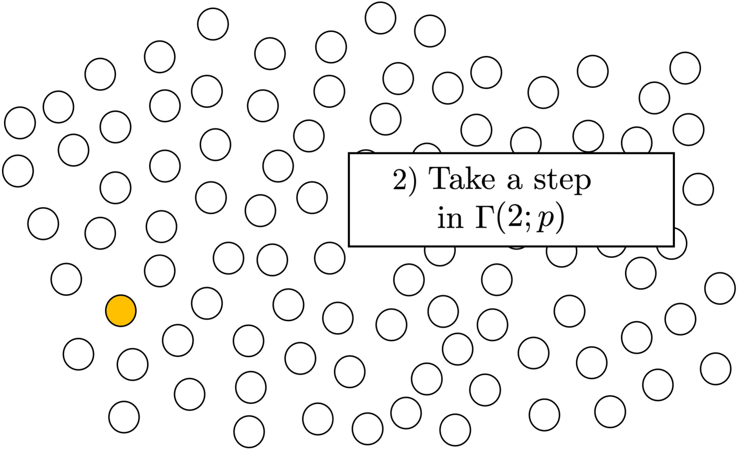

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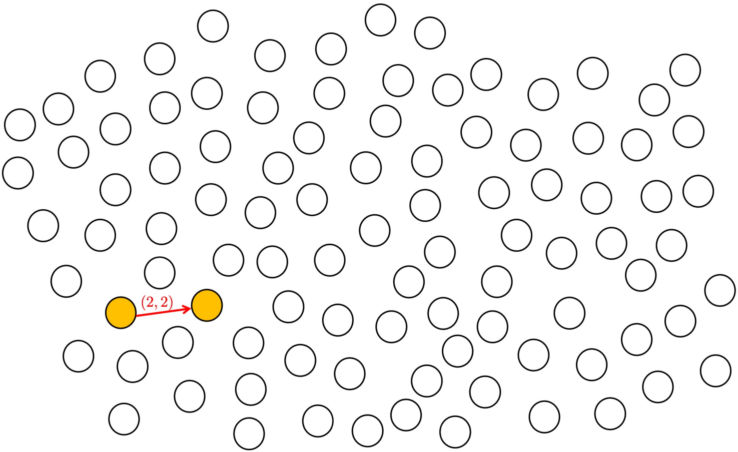

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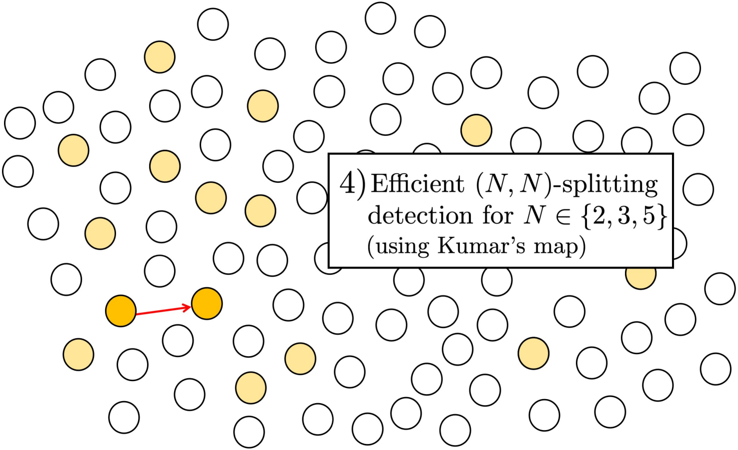

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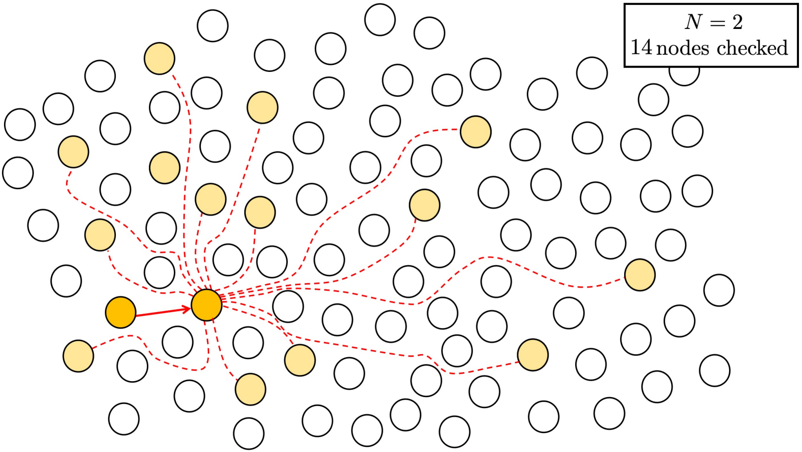

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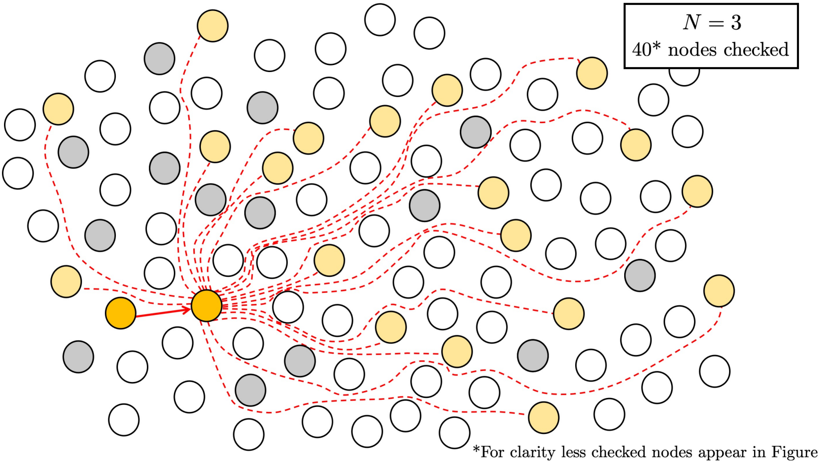

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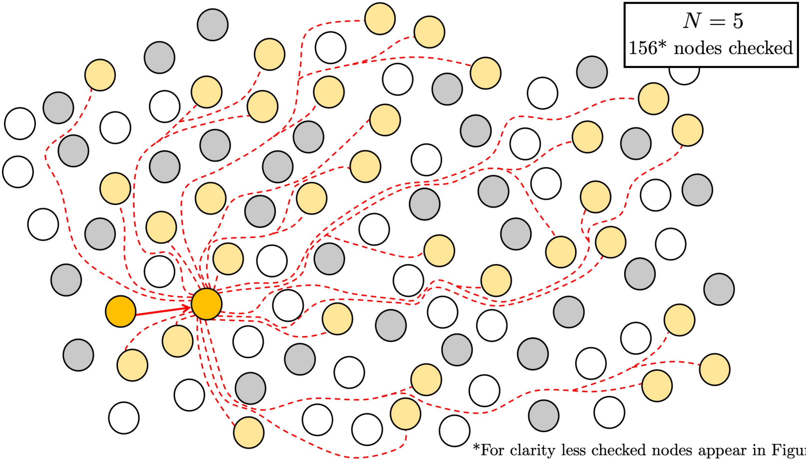

*For clarity less checked nodes appear in Figure

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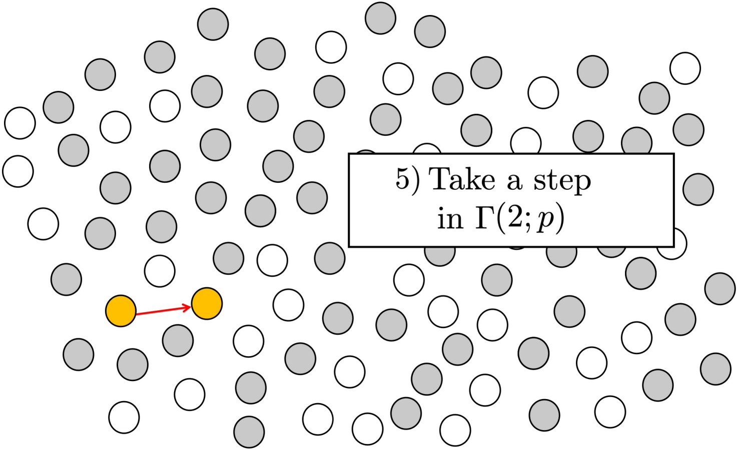

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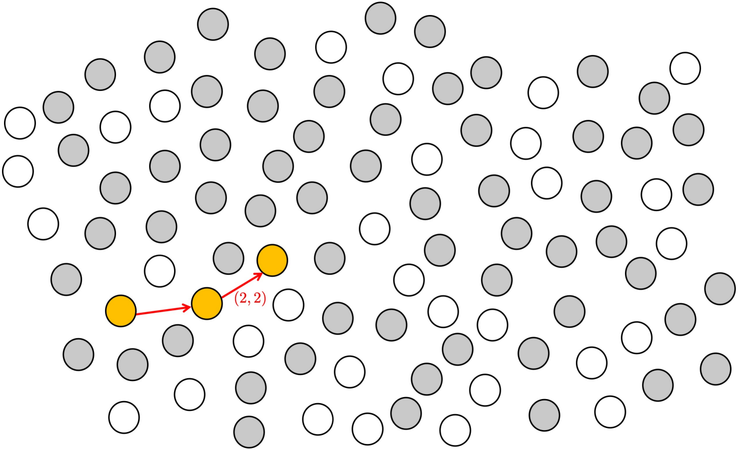

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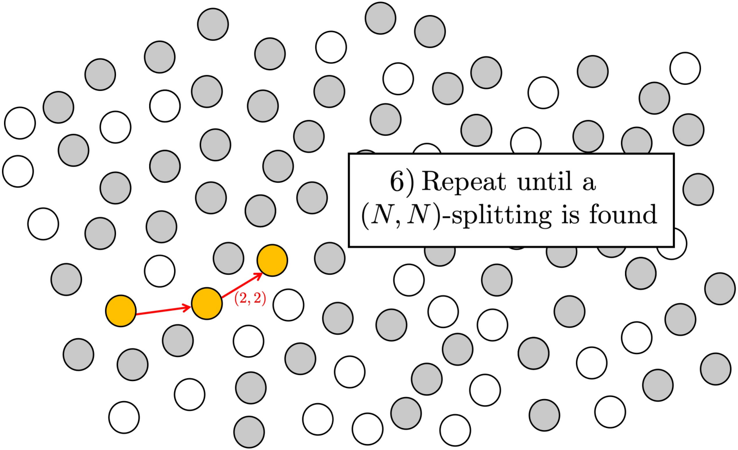

Results

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prime p	bits p	nodes per 10^8 muls	muls per node	set $N \in \{\dots\}$	nodes per 10^8 muls	muls per node	**imprv. factor**
$2^{11} \cdot 3^{24} - 1$	50	172712	579	$\{2, 3\}$	2830951	35	**16.5**
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Any questions?

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