COMPLEX DIFFERENTIAL EQUATIONS – Example Sheet 2

TKC Lent 2008

1. Find a second order linear differential equation with both $\sin z^{1/2}$ and $\cos z^{1/2}$ as solutions. Is the differential equation unique? Where are its singular points and what are the indicial equations there? Find the transition matrix for the curve $\gamma : [0, 2\pi] \to \mathbb{C};\ t \mapsto e^{it}$.
2. Find a second order linear differential equation with both $z^{1/2}$ and $z^{1/2} \log z$ as solutions. Is the differential equation unique? Where are its singular points and what are the indicial equations there? Find the transition matrix for the curve $\gamma : [0, 2\pi] \to \mathbb{C};\ t \mapsto e^{it}$.
3. Solve the differential equation:
$$z^2 f''(z) - 3zf'(z) + 4f(z) = 0 \ .$$
4. Show that the Gaussian hypergeometric differential equation:
$$z(z-1)f''(z) + [(a+b+1)z - c]f'(z) + abf(z) = 0$$
has a power series solution that begins
$$f(z) = 1 + \frac{ab}{c} z + \frac{a(a+1)\, b(b+1)}{1 \times 2\, c(c+1)} z^2 + \dots \ .$$
Find a formula for the nth coefficient when c is not an integer. What happens when c is an integer? What is the radius of convergence of the power series?
What are the singular points of the equation and the indicial equation at each?
This solution is usually denoted by $F(a, b, c; z)$ and called the *Gaussian hypergeometric function.*
5. Prove that
   (a) $\frac{dF}{dz}(a, b, c; z) = \frac{ab}{c} F(a+1, b+1, c+1; z)$.
   (b) $(1 - z)^{-a} = F(a, b, b; z)$.
   (c) $\sin^{-1} z = zF(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}; z^2)$.
6. Consider the matrix form of the Riemann hypergeometric differential equation:
$$F'(z) = \left(\frac{A}{z} + \frac{B}{z-1} \right) F(z) \ .$$
Let G be the group of those Möbius transformations that permute the three singular points $0, 1$ and ∞ in $\mathbb{P}$. Find the transformations in G explicitly and identify G as an abstract group. For each $T \in G$, show that $\widetilde{F}(z) = F(T(z))$ is a solution of another Riemann hypergeometric differential equation.
Which, if any, of the transformations in G map solutions
$$\mathcal{P} \left\{ \begin{array}{ccccc} 0 & 1 & \infty & & \\ \alpha_1 & \beta_1 & \gamma_1 & ; & z \\ \alpha_2 & \beta_2 & \gamma_2 & & \end{array} \right\}$$
of the scalar Riemann hypergeometric differential equation to other solutions?
7. Legendre's equation is:
$$(1 - z^2)f''(z) - 2zf'(z) + n(n+1)f(z) = 0 \ .$$
Where are its singular points? Show how the solutions are related to hypergeometric functions.
8. Let f be a solution of the linear differential equation:
$$f''(z) + a_1(z)f'(z) + a_0(z)f(z) = 0 \ .$$
Show that the logarithmic derivative: $g(z) = f'(z)/f(z)$ satisfies the Riccati differential equation:
$$g'(z) + a_0(z) + a_1(z)g(z) + g(z)^2 = 0 \ .$$
More generally, $g(z) = f'(z)/c(z)f(z)$ satisfies
$$g'(z) + \frac{a_0(z)}{c(z)} + \left(a_1(z) + \frac{c'(z)}{c(z)} \right) g(z) + c(z)g(z)^2 = 0 \ .$$
Use this to solve the Riccati differential equation:
$$g'(z) + b_1(z)g(z) + b_2(z)g(z)^2 = 0 \ .$$

9. Show that $g(z) = 2z/(z^2 - 1)$ is a solution of

$$g'(z) = -\frac{g(z)}{z(z^2-1)} - \tfrac{1}{2}g(z)^2 \ .$$

Show that the general solution is

$$g(z) = \frac{2z}{(z^2-1)^{1/2}[(z^2-1)^{1/2} - C]} \ .$$

Where are the singular points?

10. Show that

$$g'(z) = \frac{1}{2z} - \frac{1}{2z}g(z) + \tfrac{1}{2}g(z)^2$$

has a solution $z^{-1/2}\tan z^{1/2}$ and find the general solution.

11. Show that the product

$$g(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z}{n}\right)e^{z/n}$$

converges and satisfies

$$g'(z) = g(z)\sum_{n=1}^{\infty}\left(\frac{1}{z-n} + \frac{1}{n}\right).$$

Deduce that $g(z+1) = -zg(z)e^{\gamma}$ for some constant γ and prove that

$$\gamma = \lim_{N\to\infty}\sum_{n=1}^{N}\frac{1}{n} - \log N.$$

(This is Euler's constant.)

12. Show that a Blaschke product converges locally uniformly on $\mathbb{P} \setminus \mathbb{D}$. Where are its poles? More generally, prove that it converges on the complement of the closure of the zeros (z_n). Give an example of a Blaschke product where the unit circle is a natural boundary (so the product can not be analytically continued from $\mathbb{D}$ to any larger domain).

13. Show how to construct Weierstrass products on the unit disc in order to produce an analytic function $f : \mathbb{D} \to \mathbb{C}$ with zeros at the points (z_n) where (z_n) is any discrete set of points in $\mathbb{D}$ that does not accumulate at any point in the interior of $\mathbb{D}$.

14. Let D be a proper subdomain of the complex plane. For $z \in D$, set

$$d(z) = \inf\{|z - w| : w \in \mathbb{C} \setminus D\} \ .$$

Show that the zeros of a non-constant analytic function $f : D \to \mathbb{C}$ must be finite or else a sequence (z_n) with $d(z_n) \to 0$ as $n \to \infty$.

The following argument shows how to construct Weierstrass products to prove the converse. Let (z_n) be a sequence in D with $d(z_n) \to 0$. For each z_n chose $w_n \in \mathbb{C} \setminus D$ with $|z_n - w_n| = d(z_n)$. Show that there are polynomials P_n with

$$\left|\log\left(1 - \frac{z_n - w_n}{z - w_n}\right) - P_n\left(\frac{z_n - w_n}{z - w_n}\right)\right| \leqslant 2^{-n}$$

for $|z - w_n| \geqslant 2d(z_n)$. Hence the product

$$\prod\left(\frac{z - z_n}{z - w_n}\right)\exp -P_n\left(\frac{z_n - w_n}{z - w_n}\right)$$

converges locally uniformly on D.

15. Consider the linear differential equation:

$$f''(z) + 2p(z)f'(z) + q(z)f(z) = 0 \ .$$

Let f_1, f_2 be two linearly independent solutions. Show that the Wronskian satisfies

$$W'(z) + 2p(z)W(z) = 0$$

and deduce that $W(z) = C \exp -2P(z)$ for some constant C and a function P with $P' = p$. Prove that $g(z) = f(z) \exp P(z)$ satisfies the differential equation

$$g''(z) + I(z)g(z) = 0 \qquad \text{for} \qquad I(z) = -p'(z) - 2p(z)^2 + q(z) \ .$$

(This is the normal form of the differential equation.) What is the Wronskian for this differential equation

The *Schwarzian derivative* $\mathcal{S}u$ of an analytic function u is defined as

$$\mathcal{S}u = \left(\frac{u''}{u'}\right)' - \frac{1}{2}\left(\frac{u''}{u'}\right)^2 \ .$$

Show that $\mathcal{S}(T \circ u) = \mathcal{S}u$ for any Möbius transformation T. Find all of the functions u with $\mathcal{S}u \equiv 0$. Show that the ratio $u = f_1/f_2$ satisfies $\mathcal{S}u = 2I(z)$.