COMPLEX DIFFERENTIAL EQUATIONS – Example Sheet 2 (For supervisors.)

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1. Find a second order linear differential equation with both $\sin z^{1/2}$ and $\cos z^{1/2}$ as solutions. Is the differential equation unique? Where are its singular points and what are the indicial equations there? Find the transition matrix for the curve $\gamma : [0, 2\pi] \to \mathbb{C};\ t \mapsto e^{it}$.

$$zf''(z) + \tfrac{1}{2}f'(z) + \tfrac{1}{4}f(z) = 0$$

is unique and has regular singular points at 0 and ∞. The indicial equation is $\lambda(\lambda - \frac{1}{2})$ at both. Analytically continuing around γ changes $z^{1/2}$ to $-z^{1/2}$ and so the transition matrix, relative to the given basis, is

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{i\pi} & 0 \\ 0 & 1 \end{pmatrix} .$$

2. Find a second order linear differential equation with both $z^{1/2}$ and $z^{1/2} \log z$ as solutions. Is the differential equation unique? Where are its singular points and what are the indicial equations there? Find the transition matrix for the curve $\gamma : [0, 2\pi] \to \mathbb{C};\ t \mapsto e^{it}$.

$$z^2 f''(z) + \tfrac{1}{4}f(z) = 0$$

has regular singular points at 0 and ∞. The indicial equation is $(\lambda - \frac{1}{2})^2 = 0$. The transition matrix is

$$\begin{pmatrix} -1 & 0 \\ -2\pi i & -1 \end{pmatrix} .$$

3. Solve the differential equation:

$$z^2 f''(z) - 3zf'(z) + 4f(z) = 0 .$$

Regular singular points at 0 and ∞. Indicial equation $(\lambda - 2)^2$. One solution is z^2. Looking for others in the form $z^2 u(z)$ gives $zu''(z) + u'(z) = 0$ and so $u(z) = A \log z + B$.

4. Show that the Gaussian hypergeometric differential equation:

$$z(z-1)f''(z) + [(a+b+1)z - c]f'(z) + abf(z) = 0$$

has a power series solution that begins

$$f(z) = 1 + \frac{ab}{c}z + \frac{a(a+1)\,b(b+1)}{1 \times 2\,c(c+1)}z^2 + \dots .$$

Find a formula for the nth coefficient when c is not an integer. What happens when c is an integer? What is the radius of convergence of the power series?

What are the singular points of the equation and the indicial equation at each?

This solution is usually denoted by $F(a, b, c; z)$ and called the *Gaussian hypergeometric function.*

The singular points are $0, 1, \infty$ and the indicial equations are

$$\begin{aligned} \lambda(\lambda-1) + c\lambda &= \lambda(\lambda - 1 + c) \\ \lambda(\lambda-1) + (a+b+1-c)\lambda &= \lambda(\lambda + a + b - c) \\ \lambda(\lambda-1) - (a+b+1)\lambda + ab &= (\lambda - a)(\lambda - b) \end{aligned}$$

respectively. Hence the roots are $0, 1-c$; $0, -a-b+c$; and a, b.

5. Prove that
   (a) $\frac{dF}{dz}(a,b,c;z) = \frac{ab}{c}F(a+1,b+1,c+1;z)$.
   (b) $(1-z)^{-a} = F(a,b,b;z)$.
   (c) $\sin^{-1} z = zF(\frac{1}{2},\frac{1}{2},\frac{3}{2};z^2)$.

   (a) Differentiate the hypergeometric differential equation for $F(a,b,c;z)$.

   (b) Either show that $(1-z)^{-a}$ satisfies the hypergeometric differential equation or else show that the power series for $F(a,b,b;z)$ is a binomial series.

   (c) Write $\sin^{-1} z = z\phi(z^2)$ for an analytic function ϕ with $\phi(0) = 1$. Then compute the first and second derivatives of $\sin^{-1} z$ in terms of ϕ and use these expressions to show that ϕ satisfies the hypergeometric differential equation.

6. Consider the matrix form of the Riemann hypergeometric differential equation:
$$F'(z) = \left(\frac{A}{z} + \frac{B}{z-1}\right) F(z) \ .$$
Let G be the group of those Möbius transformations that permute the three singular points $0, 1$ and ∞ in $\mathbb{P}$. Find the transformations in G explicitly and identify G as an abstract group. For each $T \in G$, show that $\widetilde{F}(z) = F(T(z))$ is a solution of another Riemann hypergeometric differential equation.
Which, if any, of the transformations in G map solutions
$$\mathcal{P}\begin{Bmatrix} 0 & 1 & \infty & & \\ \alpha_1 & \beta_1 & \gamma_1 & ; & z \\ \alpha_2 & \beta_2 & \gamma_2 & & \end{Bmatrix}$$
of the scalar Riemann hypergeometric differential equation to other solutions?

   For a Möbius transformation T we have
$$\widetilde{F}'(z) = T'(z)F'(T(z)) = T'(z)\left(\frac{A}{T(z)} + \frac{B}{T(z)-1}\right) F(T(z)) = \left(A\frac{T'(z)}{T(z)} + B\frac{T'(z)}{T(z)-1}\right)\widetilde{F}(z)$$
   The group G is the permutation group on the three points $0, 1, \infty$. Each $T \in G$ permutes the residues at the three singularities.

   A similar result holds for the Riemann hypergeometric differential equation but not for the Gauss hypergeometric differential equation.

7. Legendre's equation is:
$$(1-z^2)f''(z) - 2zf'(z) + n(n+1)f(z) = 0 \ .$$
Where are its singular points? Show how the solutions are related to hypergeometric functions.
$$\mathcal{P}\begin{Bmatrix} -1 & 1 & \infty & & \\ 0 & 0 & -n & ; & z \\ 0 & 0 & n+1 & & \end{Bmatrix}$$

8. Let f be a solution of the linear differential equation:
$$f''(z) + a_1(z)f'(z) + a_0(z)f(z) = 0 \ .$$
Show that the logarithmic derivative: $g(z) = f'(z)/f(z)$ satisfies the Riccati differential equation:
$$g'(z) + a_0(z) + a_1(z)g(z) + g(z)^2 = 0 \ .$$
More generally, $g(z) = f'(z)/c(z)f(z)$ satisfies
$$g'(z) + \frac{a_0(z)}{c(z)} + \left(a_1(z) + \frac{c'(z)}{c(z)}\right) g(z) + c(z)g(z)^2 = 0 \ .$$
Use this to solve the Riccati differential equation:
$$g'(z) + b_1(z)g(z) + b_2(z)g(z)^2 = 0 \ .$$

   Set $a_0 = 0$, $a_1 = b_1$, $c = b_2$. Then we need to solve
$$f''(z) + b_1(z)f'(z) = 0 \ .$$

9. Show that $g(z) = 2z/(z^2 - 1)$ is a solution of

$$g'(z) = -\frac{g(z)}{z(z^2-1)} - \tfrac{1}{2}g(z)^2 \ .$$

Show that the general solution is

$$g(z) = \frac{2z}{(z^2-1)^{1/2}[(z^2-1)^{1/2} - C]} \ .$$

Where are the singular points?

Use the previous question to show that the general solution is

$$g(z) = \frac{2z}{(z^2-1)^{1/2}((z^2-1)^{1/2} - C)} \ .$$

10. Show that

$$g'(z) = \frac{1}{2z} - \frac{1}{2z}g(z) + \tfrac{1}{2}g(z)^2$$

has a solution $z^{-1/2}\tan z^{1/2}$ and find the general solution.

Look for a solution of the form $z^{-1/2}h(z)$ *and show that*

$$h'(z) = \frac{1}{2z^{1/2}}(1 + h(z)^2) \ .$$

So the general solution is

$$\tan(z^{1/2} + C)/z^{1/2} \ .$$

11. Show that the product

$$g(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z}{n}\right)e^{z/n}$$

converges and satisfies

$$g'(z) = g(z)\sum_{n=1}^{\infty}\left(\frac{1}{z-n} + \frac{1}{n}\right).$$

Deduce that $g(z+1) = -zg(z)e^{\gamma}$ for some constant γ and prove that

$$\gamma = \lim_{N\to\infty}\sum_{n=1}^{N}\frac{1}{n} - \log N.$$

(This is Euler's constant.)

The product converges by comparison with $\sum \frac{1}{n^2}$. *Then* $\log g(z)$ *is a convergent sum and we can differentiate it term by term.*

$$g(z+1) = -zg(z)\prod_{n=1}^{\infty}\frac{n}{n+1}e^{1/n} \ .$$

Compare the Γ*-function, which has simple poles at all the points* $0, -1, -2, \ldots$.

12. Show that a Blaschke product converges locally uniformly on $\mathbb{P}\setminus\mathbb{D}$. Where are its poles? More generally, prove that it converges on the complement of the closure of the zeros (z_n). Give an example of a Blaschke product where the unit circle is a natural boundary (so the product can not be analytically continued from $\mathbb{D}$ to any larger domain).

Observe that

$$\frac{|z_n|}{-z_n}\left(\frac{z - z_n}{1 - \overline{z_n}z}\right) - 1 = \frac{1-|z_n|}{-z_n}\left(\frac{|z_n|z + z_n}{1 - \overline{z_n}z}\right) \ .$$

13. Show how to construct Weierstrass products on the unit disc in order to produce an analytic function $f : \mathbb{D} \to \mathbb{C}$ with zeros at the points (z_n) where (z_n) is any discrete set of points in $\mathbb{D}$ that does not accumulate at any point in the interior of $\mathbb{D}$.

14. Let D be a proper subdomain of the complex plane. For $z \in D$, set

$$d(z) = \inf\{|z - w| : w \in \mathbb{C} \setminus D\} .$$

Show that the zeros of a non-constant analytic function $f : D \to \mathbb{C}$ must be finite or else a sequence (z_n) with $d(z_n) \to 0$ as $n \to \infty$.

The following argument shows how to construct Weierstrass products to prove the converse. Let (z_n) be a sequence in D with $d(z_n) \to 0$. For each z_n chose $w_n \in \mathbb{C} \setminus D$ with $|z_n - w_n| = d(z_n)$. Show that there are polynomials P_n with

$$\left| \log \left(1 - \frac{z_n - w_n}{z - w_n} \right) - P_n \left(\frac{z_n - w_n}{z - w_n} \right) \right| \leqslant 2^{-n}$$

for $|z - w_n| \geqslant 2d(z_n)$. Hence the product

$$\prod \left(\frac{z - z_n}{z - w_n} \right) \exp -P_n \left(\frac{z_n - w_n}{z - w_n} \right)$$

converges locally uniformly on D.

Choose P_n to be a partial sum of the Taylor series for $\log(1 - \zeta)$ with $|\log(1 - \zeta) - P_n(\zeta)| \leqslant 2^{-n}$ for $|\zeta| \leqslant \frac{1}{2}$.

If $|z - w_n| \geqslant 2d(z_n) = 2|z_n - w_n|$, then

$$\zeta = \frac{z_n - w_n}{z - w_n} \qquad \textit{satisfies} \qquad |zeta| \leqslant \tfrac{1}{2} .$$

Since $d(z_n) \to 0$, we have $|z - w_n| \geqslant d(z) \geqslant 2d(z_n)$ for all sufficiently large n.

15. Consider the linear differential equation:

$$f''(z) + 2p(z)f'(z) + q(z)f(z) = 0 .$$

Let f_1, f_2 be two linearly independent solutions. Show that the Wronskian satisfies

$$W'(z) + 2p(z)W(z) = 0$$

and deduce that $W(z) = C \exp -2P(z)$ for some constant C and a function P with $P' = p$. Prove that $g(z) = f(z) \exp P(z)$ satisfies the differential equation

$$g''(z) + I(z)g(z) = 0 \qquad \text{for} \qquad I(z) = -p'(z) - 2p(z)^2 + q(z) .$$

(This is the normal form of the differential equation.) What is the Wronskian for this differential equation

The *Schwarzian derivative* $\mathcal{S}u$ of an analytic function u is defined as

$$\mathcal{S}u = \left(\frac{u''}{u'} \right)' - \frac{1}{2} \left(\frac{u''}{u'} \right)^2 .$$

Show that $\mathcal{S}(T \circ u) = \mathcal{S}u$ for any Möbius transformation T. Find all of the functions u with $\mathcal{S}u \equiv 0$. Show that the ratio $u = f_1/f_2$ satisfies $\mathcal{S}u = 2I(z)$.

To solve $\mathcal{S}u \equiv 0$, set $r = u''/u'$. Then we have the simple Riccati differential equation:

$$r' = \tfrac{1}{2} r^2 .$$

Solutions are

$$r = \frac{-2}{z + c} \qquad u(z) = \frac{Az + B}{z + c}$$

(and $r \equiv 0$).

Let $u = f_1/f_2$, so $u' = W/f_2^2$. Then

$$\frac{u''}{u'} = (\log u')' = \frac{W'}{W} - 2\frac{f_2'}{f_2} = -2 \left(p + \frac{f_2'}{f_2} \right) .$$

Therefore,

$$\mathcal{S}u = -2 \left(p + \frac{f_2'}{f_2} \right)' - 2 \left(p + \frac{f_2'}{f_2} \right)^2 = -2p' - 2p^2 + 2q .$$