COMPLEX DIFFERENTIAL EQUATIONS – Example Sheet 3 (For supervisors.)

TKC Lent 2008

1. Show that each of the following equations has a fixed singularity, where, along a suitable path approaching the singularity, the solutions have no limits.

$$f'(z) = z^{-2} f(z)$$

$$f'(z) = i(1-z)^{-1} f(z)$$

$$f'(z) = f(z) \ .$$

Solve by separation of variables.

$f(z) = Ae^{1/z}$ *singularity at* 0 *and* $f(iy)$ *has no limit as* $y \to 0$.

$f(z) = A(1-z)^{-i}$ *singularity at* 1 *and* $f(1-e^{-t})$ *has no limit as* $t \to +\infty$.

$f(z) = Ae^{z}$ *singularity at* ∞ *and* $f(iy)$ *has no limit as* $y \to 0$.

Note, in each case, that $A = 0$ *gives a non-singular solution.*

2. Give an example of a singular point of a differential equation where there is at least one solution that is analytic at that point.

See question 1. Alternatively, construct a second order linear differential equation with solutions, say, z *and* $z^{1/2}$. ($2z^2 f''(z) - zf'(z) + f(z) = 0$)

3. Find all of the fixed singularities of

$$(z + f(z))f'(z) - z + f(z) = 0$$

and determine the character of the solutions near these points. Show that there are movable branch points of order 1.

Write $w = f(z)$, *so*

$$\frac{dw}{dz} = \frac{z-w}{z+w} = \frac{P(z,w)}{Q(z,w)} \ .$$

Possible fixed singularities are at points z_o *where:*

(a) $Q(z_o, \cdot) \equiv 0$.

(b) *There exists* w_o *with* $P(z_o, w_o) = Q(z_o, w_o) = 0$.

(c) *Write* $\omega = 1/w$, *so*

$$\frac{d\omega}{dz} = -\omega^2 \frac{P(z, 1/\omega)}{Q(z, 1/\omega)} = \frac{P_1(z,\omega)}{Q_1(z,\omega)}$$

for polynomials P_1, Q_1. *There exists* ω_o *with* $P_1(z_o, \omega_o) = Q_1(z_o, \omega_o) = 0$.

For this example, $P(z,w) = z - w$, $Q(z,w) = z + w$. *So there are no fixed singularities of type (a). For (b) we have* $z_0 = 0$ *or* ∞. *For (c),* $P_1(z,\omega) = \omega^2(1 - z\omega)$, $Q_1(z,\omega) = 1 + z\omega$. *So there are no fixed singularities of type (c). In fact even* 0 *and* ∞ *are not singularities.*

The equation is homogeneous so we solve it by setting $w = zv$. *Then*

$$z\frac{dv}{dz} + v = \frac{1-v}{1+v} \ .$$

Separating the variables gives $(v+1)^2 - 2 = Az^{-2}$, *so*

$$f(z) = z\left((2 + Az^{-2})^{1/2} - 1\right) \ .$$

This is meromorphic at both 0 *and* ∞.

There are movable branch points where $Q(z_o, w_o) = 0$. *Now* $Q(z_o, w) = z_o + w$ *has a simple zero at* $-z_o$ *so the branch points are of order* 1. *These are the points where the square root* $(2 + Az^{-2})^{1/2}$ *is singular* ($A = -2z_o^2$).

4. Find the fixed singular points of

$$f'(z) = P(z, f(z))$$

where P is a polynomial in 2 variables.

There are no fixed singular points of type (a) or (b). Write $P(z,w) = \sum_{n=0}^{N} P_n(z)w^n$ with $P_N \not\equiv 0$. Then

$$\frac{d\omega}{dz} = -\frac{\sum_{n=0}^{N} P_n(z)\omega^{N-n}}{\omega^{N-2}} \ .$$

So there are fixed singular points at z_o where $P_1(z_o,\cdot)$ and $Q_1(z_o,\cdot)$ have a common zero. This is where $P_N(z_o) = 0$.

5. Find the singularities of

$$f'(z) = z^{1/2} + z^{3/2}f(z) - f(z)^2 \ .$$

The coefficients are algebraic so we need to add to the possible fixed singularities listed in the answer to question 3 the singularities of the coefficients. These are 0 and ∞. By question 4, there are no other fixed singularities.

[We can convert this to a differential equation with holomorphic coefficients by setting $x = z^{1/2}$. Then

$$\frac{dw}{dz} = 2x^2 + 2x^4w - 2xw^2$$

is a Riccati equation and has local power series solutions. Thus the solutions of the original equation are power series in $z^{1/2}$]

6. Show that

$$f'(z) = z^3 + f(z)^3 \ ; \quad f(0) = w_0$$

has movable branch points and find their order. If $w_o > 0$, the branch point $b(w_o)$ nearest to the origin lies on the positive real axis. How does $b(w_o)$ change as w_o increases? Where are the fixed singular points of the differential equation, if any?

By question 4 the differential equation

$$\frac{dw}{dz} = z^3 + w^3$$

has no fixed singularities. At points z_o where $w(z_o)$ is finite, the solution is locally holomorphic. Now consider those points z_o where $w(z_o) = \infty$.

Write $\omega = 1/w$ to get

$$\frac{d\omega}{dz} = \frac{1 + z^3\omega^3}{-\omega} \ .$$

Note the pole where $\omega = 0$. To solve this near $z = z_o$, $w = \infty$, $\omega = 0$, write it as

$$\frac{dz}{d\omega} = \frac{-\omega}{1 + z^3\omega^3} \ .$$

This has a power series solution:

$$z = z_0 - \tfrac{1}{2}\omega^2\left(1 + a_1\omega + a_2\omega^2 + \ldots\right) = z_0 - \tfrac{1}{2}(\omega h(\omega))^2 \ .$$

So $\omega h(\omega) = -2(z - z_o)^{1/2}$ and hence ω is a power series in $(z - z_0^{1/2}$, say $\omega = b_1(z - z_o) + b_2(z - z_o)^2 + \ldots$. Now it is clear that $w = 1/\omega$ has a branch point at z_o of order 1

Consider only non-negative real values for z and w. The graph of w against z is strictly increasing on $[0, b(w_o))$ and tends to $+\infty$ at $b(w_o)$. Now the solutions for different values of w_o can not intersect, so, as w_o increases, so the point $b(w_o)$ where w becomes infinite must decrease.