If (z_n) is an infinite sequence of points in $\mathbb{C}$ which converges to ∞ then the product

$$\prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right)$$

need not converge. However, if $\sum |z_n|^{-1}$ converges, then the product will converge to an entire function with zeros precisely at the points z_n. To deal with sequences (z_n) which have $\sum |z_n|^{-1}$ divergent we need to introduce exponential factors into the product.

Theorem Weierstrass products

Let (z_n) be a sequence of points in $\mathbb{C}$ which is either finite or else tends to ∞. Then there is an entire function f which has a zero at each point ζ in the sequence with order equal to the number of times that it occurs in the sequence, and no other zeros. If g is another such function then $f(z) = g(z) \exp h(z)$ for some entire function h.

Proof:

Choose positive numbers M_n for which $\sum M_n$ converges. The function $z \mapsto \mathrm{Log}\left(1 - \frac{z}{z_n}\right)$ is analytic on $\{z : |z| < |z_n|\}$ so its Taylor series

$$-\frac{z}{z_n} - \frac{1}{2}\left(\frac{z}{z_n}\right)^2 - \frac{1}{3}\left(\frac{z}{z_n}\right)^3 - \dots$$

converges uniformly on $\{z : |z| \leqslant \frac{1}{2}|z_n|\}$. Hence we can choose natural numbers $N(n)$ so that

$$q_n(z) = \frac{z}{z_n} + \frac{1}{2}\left(\frac{z}{z_n}\right)^2 + \frac{1}{3}\left(\frac{z}{z_n}\right)^3 + \dots + \frac{1}{N(n)}\left(\frac{z}{z_n}\right)^{N(n)}$$

satisfies

$$\left|\mathrm{Log}\left(1 - \frac{z}{z_n}\right) + q_n(z)\right| \leqslant M_n \qquad \text{for} \qquad |z| \leqslant \tfrac{1}{2}|z_n|.$$

Therefore, the series

$$\sum_{n=1}^{\infty} \left(\mathrm{Log}\left(1 - \frac{z}{z_n}\right) + q_n(z)\right)$$

will converge locally uniformly. Hence,

$$f(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) \exp q_n(z)$$

converges and gives an entire function f with the desired properties.

If g were another such function then g/f would be an entire function with no zeros and therefore equal to $\exp h$ for some entire function h. $\square$

Corollary

Every meromorphic function $f : \mathbb{C} \to \mathbb{C}_\infty$ is the quotient a/b of two entire functions a and b.

Proof:

The theorem enables us to construct an entire function b whose zeros are poles of f. Then $a = b.f$ is also entire. □

As an example, let us try to construct a entire function with zeros at the integer points. The series $\sum n^{-2}$ converges so the proof of Weierstrass theorem shows that

$$f(z) = z \prod_{n \neq 0} \left(1 - \frac{z}{n}\right) e^{z/n}$$

converges to the desired entire function. We can rewrite this series as

$$f(z) = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

Because of the locally uniform convergence we can differentiate the product to obtain

$$\begin{aligned} f'(z) &= f(z) \left\{ \frac{1}{z} + \sum_{n \neq 0} \left(\frac{1}{z-n} + \frac{1}{n} \right) \right\} \\ &= f(z) \left\{ \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{2z}{z^2 - n^2} \right) \right\} \end{aligned}$$

Hence $f'(z) = f(z)\varepsilon_1(z) = f(z)\pi \cot \pi z$. We also have $f'(0) = 1$ so we can solve this differential equation to obtain

$$z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) = f(z) = \frac{\sin \pi z}{\pi}.$$

Exercises

Show that the product

$$g(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{n}\right) e^{z/n}$$

converges and satisfies

$$g'(z) = g(z) \sum_{n=1}^{\infty} \left(\frac{1}{z-n} + \frac{1}{n} \right).$$

Deduce that $g(z+1) = -zg(z)e^{\gamma}$ for some constant γ and prove that

$$\gamma = \lim_{N \to \infty} \sum_{n=1}^{N} \frac{1}{n} - \log N.$$

(This is Euler's constant.)
