GEOMETRY — Example Sheet 2

TKC Lent 2009

1. Prove that two points $w, z \in \mathbb{C}_\infty$ correspond to antipodal points in S^2 under stereographic projection if, and only if, $w = J(z)$ for the transformation $J(z) = -1/\overline{z}$.

   Show that any Möbius transformation T other than the identity has either one or two fixed points on $\mathbb{C} \cup \{\infty\}$. Show that the Möbius transformation corresponding under stereographic projection to a non-trivial rotation has two antipodal fixed points.

   Show that a Möbius transformation $T : z \mapsto (az+b)/(cz+d)$ with $ad - bc = 1$ satisfies $J^{-1}TJ = T$ precisely when $d = \overline{a}$ and $c = -\overline{b}$.

2. Prove that Möbius transformations of the extended complex plane $\mathbb{C}_\infty$ preserve cross-ratios. Let the points $u, v \in \mathbb{C}$ correspond under stereographic projection to points $\boldsymbol{P}, \boldsymbol{Q} \in S^2$. Show that the cross-ratio of the four points $u, v, -1/\overline{u}, -1/\overline{v}$ (in some order) is equal to $-\tan^2 \frac{1}{2} d(\boldsymbol{P}, \boldsymbol{Q})$, where $d(\boldsymbol{P}, \boldsymbol{Q})$ is the spherical distance between $\boldsymbol{P}$ and $\boldsymbol{Q}$.

3. Let $J : z \mapsto 1/\overline{z}$ be inversion in the unit circle and recall that Möbius transformations map inverse points to inverse points.

   Show that, a Möbius transformation T maps the unit circle onto itself if and only if $J^{-1}TJ = T$. Deduce that a Möbius transformation

   $$T : z \mapsto \frac{az+b}{cz+d} \qquad \text{with} \qquad ad - bc = 1$$

   maps the unit disc $\mathbb{D}$ onto itself if and only if $d = \overline{a}$ and $c = \overline{b}$. Show that every such transformation is an isometry for the hyperbolic metric.

   Show that we can also write these Möbius transformations as

   $$z \mapsto \zeta \left(\frac{z - z_o}{1 - \overline{z_o} z} \right)$$

   for some $z_o \in \mathbb{D}$ and some $\zeta \in \mathbb{C}$ of modulus 1.

4. Let Γ be the hyperbolic circle $\{z \in \mathbb{D} : \rho(z, z_0) = \rho_o\}$ in the disc $\mathbb{D}$. Show that it is also an Euclidean circle and a spherical circle but that the Euclidean or spherical centre and radius can be different from the hyperbolic centre z_o and radius ρ_o.

5. Show that a hyperbolic circle with <u>hyperbolic</u> radius r has length $2\pi \sinh r$ and encloses a disc of hyperbolic area $4\pi \sinh^2 \frac{1}{2} r$. Sketch these as functions of r.

6. Show that two hyperbolic lines have a common orthogonal line if and only if they are ultraparallel. Prove that, in this case, the common orthogonal line is unique.

7. Fix a point P on the boundary of the unit disc $\mathbb{D}$. Describe the curves in $\mathbb{D}$ that are orthogonal to every hyperbolic line that passes through P.

8. Prove that a hyperbolic N-gon with interior angles $\alpha_1, \alpha_2, \ldots, \alpha_N$ has area $(N-2)\pi - \sum \alpha_j$. Show that, for every $N \geqslant 3$ and every α with $0 < \alpha < (1 - \frac{2}{N})\pi$, there is a regular N-gon with all angles equal to α.

9. Show that in a spherical, Euclidean or hyperbolic triangle, the angle bisectors are lines and they meet at a point.

10. Let ℓ and m be two fixed hyperbolic lines that cross at an angle α at a point $\boldsymbol{A}$. Another line n crosses ℓ at a (movable) point $\boldsymbol{B}$ and a fixed angle β. If n also crosses m at an angle θ, show that θ varies monotonically as the point $\boldsymbol{B}$ moves along the line ℓ.

    Deduce that there is a hyperbolic triangle with angles α, β, γ provided that $\alpha + \beta + \gamma < \pi$.

11. State the sine rule for hyperbolic triangles. Show that $a \leqslant b \leqslant c$ if and only if $\alpha \leqslant \beta \leqslant \gamma$.

12. If w, z are points in the upper half-plane, prove that the hyperbolic distance between them is $2 \tanh^{-1} |(w - z)/(w - \overline{z})|$.

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13. *In this question we will show how to deduce the sine rule and second cosine rule for a hyperbolic triangle from the first cosine rule.*

Use the cosine rule to show that

$$\cos\alpha = \frac{\cosh b\cosh c - \cosh a}{\sqrt{\cosh^2 b - 1}\sqrt{\cosh^2 c - 1}} \qquad \text{and} \qquad \sin^2\alpha = \frac{D^2}{(\cosh^2 b - 1)(\cosh^2 c - 1)}$$

where $D^2 = 1 - \cosh^2 a - \cosh^2 b - \cosh^2 c + 2\cosh a\cosh b\cosh c$. Deduce that

$$\frac{\sin^2\alpha}{\sinh^2 a} = \frac{D^2}{(\cosh^2 a - 1)(\cosh^2 b - 1)(\cosh^2 c - 1)} \ .$$

Show that, since the right hand side is symmetric in a, b, c, this implies the hyperbolic sine rule.

In a similar way, show that

$$\cos\beta\cos\gamma + \cos\alpha = \frac{D^2\cosh a}{(\cosh^2 a - 1)\sqrt{\cosh^2 b - 1}\sqrt{\cosh^2 c - 1}}$$

and deduce the second cosine rule:

$$\cos\alpha = -\cos\beta\cos\gamma + \sin\beta\sin\gamma\cosh a \ .$$

Deduce that two hyperbolic triangles are congruent if and only if they have the same angles.

14. Let Δ be a triangle on a sphere of radius R, with angles α, β, γ and sides of length a, b, c. Prove a version of the cosine and sine rules for this triangle.

Show that, if we formally set R equal to the complex number i, then we obtain the hyperbolic cosine and sine rules. (Thus hyperbolic geometry is the geometry of a sphere with radius i and curvature $R^2 = -1$.)

15. The *quaternions* $\mathcal{Q}$ consist of all 2×2 complex matrices

$$q = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}$$

with addition and multiplication as for the matrices. Every such quaternion q can be written as $q_0\mathbf{1} + q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k}$ where

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ ; \ \boldsymbol{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \ ; \ \boldsymbol{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ ; \ \boldsymbol{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \ .$$

Show that these four elements, together with their additive inverses $-\mathbf{1}, -\boldsymbol{i}, -\boldsymbol{j}, -\boldsymbol{k}$ form a non-commutative group: the *Quaternion* 8*-group*. We can identify the subspace of $\mathcal{Q}$ spanned by $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ with $\mathbb{R}^3$ by making $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ correspond to the standard basis vectors of $\mathbb{R}^3$. We can then write any quaternion q as $q_0\mathbf{1} + \boldsymbol{v}$ for a scalar q_0 and a vector $\boldsymbol{v} \in \mathbb{R}^3$. Prove that we then have

$$(p_0\mathbf{1} + \boldsymbol{u})(q_0\mathbf{1} + \boldsymbol{v}) = (p_0q_0 - \boldsymbol{u}\cdot\boldsymbol{v})\mathbf{1} \ + \ (p_0\boldsymbol{v} + q_0\boldsymbol{u}) \ + \ (\boldsymbol{u}\times\boldsymbol{v}) \ .$$

In particular, for two vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^3$ we have $\boldsymbol{uv} + \boldsymbol{vu} = -2(\boldsymbol{u}\cdot\boldsymbol{v})\mathbf{1}$.

The *conjugate* of a quaternion $q = q_o\mathbf{1} + \boldsymbol{v}$ is $\overline{q} = q_0\mathbf{1} - \boldsymbol{v}$. Show that $q\overline{q} = ||q||^2\mathbf{1} = \overline{q}q$ where $||q||^2 = q_0^2 + ||\boldsymbol{v}||^2$. Prove that, for any unit vector $\boldsymbol{u} \in \mathbb{R}^3$, we have

$$\boldsymbol{uxu} = \boldsymbol{x} - 2(\boldsymbol{x}\cdot\boldsymbol{u})\boldsymbol{u} \ .$$

So the map $T_{\boldsymbol{u}} : \mathbb{R}^3 \to \mathbb{R}^3 \ ; \ \ \boldsymbol{x} \mapsto \boldsymbol{uxu}$ is reflection in the plane perpendicular to $\boldsymbol{u}$. By writing any isometry of S^2 as a composite of reflection, or otherwise, show that for each quaternion q with $||q|| = 1$ the map

$$T_q : \mathbb{R}^3 \to \mathbb{R}^3 \ ; \ \ \boldsymbol{x} \mapsto q\boldsymbol{x}\overline{q}$$

is an orientation preserving isometry of S^2. Hence show that

$$T : S(\mathcal{Q}) \to \mathrm{SO}(3) \ ; \ \ q \mapsto T_q$$

is a group homomorphism from the unit sphere $S(\mathcal{Q})$ (which is a 3-dimensional sphere S^3) onto $\mathrm{SO}(3)$ with kernel $\{-\mathbf{1}, \mathbf{1}\}$.

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Supervisors can obtain an annotated version of this example sheet from DPMMS.