GEOMETRY — Example Sheet 3

TKC Lent 2009

1. Let U be an open subset of $\mathbb{R}^2$ with the Riemannian metric

$$ds^2 = E\,dx_1^2 + 2F\,dx_1\,dx_2 + G\,dx_2^2 \ .$$

For any point $P \in U$, show that there is a $\lambda > 0$ and a neighbourhood N of P with

$$(E - \lambda)\,dx_1^2 + 2F\,dx_1\,dx_2 + (G - \lambda)\,dx_2^2$$

a Riemannian metric on N.
[*Hint: A real matrix* $\begin{pmatrix} a & b \\ b & d \end{pmatrix}$ *is positive definite if and only if* $a > 0$ *and* $ad - b^2 > 0$.]
If U is path-connected, we define the distance between two points of U as the infimum of the lengths of all curves in U between those two points. Give an example where this distance is not realised as the length of any curve in U between the two points.

2. Consider the Riemannian metric

$$ds^2 = \frac{dx_1^2 + dx_2^2}{1 - (x_1^2 + x_2^2)}$$

on the unit disc $\mathbb{D}$. Prove that diameters of the disc are length minimising curves and hence geodesics. Show that the distance between points is bounded but areas are unbounded.

3. Let $U = \{(x_1, x_2) \in \mathbb{R}^2 : |x_1|, |x_2| < 1\}$ and consider the two Riemannian metrics

$$\frac{dx_1^2}{(1 - x_1^2)^2} + \frac{dx_2^2}{(1 - x_2^2)^2} \qquad \text{and} \qquad \frac{dx_1^2}{(1 - x_2^2)^2} + \frac{dx_2^2}{(1 - x_1^2)^2}$$

on U. Prove that there is no isometry between the two spaces but that an area preserving diffeomorphism does exist.
[*Consider the length of curves going out to the boundary.*]

4. For the unit sphere S in $\mathbb{R}^3$, find the unit normal at a point $\boldsymbol{x}$, the tangent plane at $\boldsymbol{x}$ and the intersection of planes parallel to the tangent plane with S.

5. Show that

$$\boldsymbol{r} : (0, \pi) \times (0, 2\pi) \to \mathbb{R}^3 \ ; \quad (u, v) \mapsto (\sin u \cos v, \sin u \sin v, \cos u)$$

is a surface parametrisation. Describe the image. What is the corresponding Riemannian metric?

6. Let T denote the torus obtained by rotating the circle $\{(x, 0, z) \in \mathbb{R}^3 : (x - 2)^2 + z^2 = 1\}$ about the z-axis. Describe a surface parametrisation for T and hence calculate its area.

7. Prove directly that the hyperbolic lines satisfy the differential equations for geodesics in the hyperbolic plane.

8. For $a > 0$, let $C(a)$ be the cone:

$$C(a) = \{(x, y, z) \in \mathbb{R}^3 : z^2 = a(x^2 + y^2) \text{ and } z > 0\} \ .$$

Find a parametrisation for $C(a)$ and hence find the geodesics on $C(a)$.
When $a = 3$, show that no (infinite) geodesic intersects itself. When $a > 3$, show that there are geodesics that intersect themselves.

9. Let $\sigma = (\sigma_1, \sigma_2) : (a, b) \to \{(x, y) \in \mathbb{R}^2 : y > 0\}$ be a unit speed curve in the upper half-plane that does not intersect itself and maps the open interval (a, b) homeomorphically onto its image. The *surface of revolution* R is then obtained by rotating σ about the x-axis. Show that

$$(s, t) \mapsto (\sigma_1(t), \sigma_2(t) \cos s, \sigma_2(t) \sin s)$$

is a surface parametrisation for part of R. Calculate the Riemannian metric and the second fundamental form. Hence show that the Gaussian curvature is given by

$$K = -\frac{\sigma_2''(t)}{\sigma_2(t)} \ .$$

10. Using the formulae from the previous question, calculate the Gaussian curvature for a sphere, for the hyperboloid of one sheet:

$$x^2 + y^2 - z^2 = +1$$

and the hyperboloid of two sheets:

$$x^2 + y^2 - z^2 = -1 \ .$$

For the torus described in question 6, mark the points where the Gaussian curvature K satisfies $K < 0$; $K = 0$ and $K > 0$.

11. Let R be a surface in $\mathbb{R}^3$ that is closed and bounded. Explain why there is a point Q of R at a maximal distance d from the origin. By considering the sphere S centred on the origin and of radius d, or otherwise, show that the Gaussian curvature of R is strictly positive at Q. Hence the closed and bounded surface R can not have Gaussian curvature less than or equal to 0 at every point.

12. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function with

$$f(0,0) = 0 \ , \quad \frac{\partial f}{\partial x}(0,0) = 0 \ , \quad \frac{\partial f}{\partial y}(0,0) = 0 \ .$$

Let $\boldsymbol{r}$ be the surface parametrisation:

$$\boldsymbol{r} : (x,y) \mapsto (x, y, f(x,y)) \ .$$

Show that the Riemannian metric at the origin is $ds^2 = dx^2 + dy^2$ and the second fundamental form is

$$\frac{\partial^2 f}{\partial x^2}\, dx^2 + 2\frac{\partial^2 f}{\partial x\, \partial y}\, dx\, dy + \frac{\partial^2 f}{\partial y^2}\, dy^2$$

(for a suitable choice of the unit normal) where all of the partial derivatives are evaluated at $(0,0)$. Deduce that the Gaussian curvature at the origin is

$$K = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial y^2}\right)^2 \ .$$

Now suppose that $g : \mathbb{R}^2 \to \mathbb{R}$ is another smooth function with $g(0,0) = 0$ and $g(x,y) \geqslant f(x,y)$ for every $(x,y) \in \mathbb{R}^2$. Show that

$$\frac{\partial g}{\partial x}(0,0) = 0 \ , \quad \frac{\partial g}{\partial y}(0,0) = 0 \ .$$

Show further that

$$\frac{\partial^2 g}{\partial x^2}\, u^2 + 2\frac{\partial^2 g}{\partial x\, \partial y}\, uv + \frac{\partial^2 g}{\partial y^2}\, v^2 \geqslant \frac{\partial^2 f}{\partial x^2}\, u^2 + 2\frac{\partial^2 f}{\partial x\, \partial y}\, uv + \frac{\partial^2 f}{\partial y^2}\, v^2$$

at $(0,0)$ and deduce that

$$\begin{pmatrix} \frac{\partial^2 g}{\partial x^2} & \frac{\partial^2 g}{\partial x\, \partial y} \\ \frac{\partial^2 g}{\partial x\, \partial y} & \frac{\partial^2 g}{\partial y^2} \end{pmatrix} \geqslant \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x\, \partial y} \\ \frac{\partial^2 f}{\partial x\, \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

at $(0,0)$.

Does this imply that the Gaussian curvature of the graph of g at the origin is greater than or equal to the Gaussian curvature of the graph of f at the origin.

Please send any comment or corrections to t.k.carne@dpmms.cam.ac.uk .

Supervisors can obtain an annotated version of this example sheet from DPMMS.