GEOMETRY

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16 **LECTURES**

An introduction to the geometry of the sphere, Euclidean plane and hyperbolic plane.

APPROPRIATE BOOKS

P.M.H. Wilson, *Curved Spaces*, CUP, 2008.

M. Do Carmo, *Differential Geometry of Curves and Surfaces*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1976.

A. Pressley, *Elementary Differential Geometry*, Springer Undergraduate Mathematics Series, Springer-Verlag, 2001.

E. Rees, *Notes on Geometry*, Springer Universitext, 1998.

M. Reid and B. Szendroi, *Geometry and Topology*, CUP, 2005.

H.S.M. Coxeter, *Introduction to Geometry*, 2nd Edition, Wiley Classics, 1989.

V.V. Nikulin & I. Shaferevich, *Geometries and Groups*, Springer Verlag, 1987.

HISTORY OF GEOMETRY

Euclid of Alexandria (*c* 300BC)

Importance of Geometry

Topology, Physics, Algebra.

The Parallel Postulate

Straight lines.

Elliptic, Euclidean and Hyperbolic planes.

Klein's Erlangen Programme

Symmetry Groups

Isometries and Invariants.

The Platonic Solids

Finite Symmetry Groups.

1 EUCLIDEAN GEOMETRY

$\mathbb{R}^N$ as a model.

Inner product $\boldsymbol{x} \cdot \boldsymbol{y} = \sum_{n=1}^{N} x_n y_n$

Norm $||\boldsymbol{x}|| = (\boldsymbol{x} \cdot \boldsymbol{x})^{1/2}$.

1.1 Euclidean N-space: $\mathbb{E}^N$.

Points of $\mathbb{E}^N$ are the elements of $\mathbb{R}^N$,

Lines in $\mathbb{E}^N$ are the sets

$$\{\boldsymbol{x} : \boldsymbol{x} = \boldsymbol{A} + \lambda \boldsymbol{u} \text{ for some } \lambda \in \mathbb{R}\}$$

with $\boldsymbol{u}$ a non-zero vector.

Euclidean Metric

$d(\boldsymbol{x}, \boldsymbol{y}) = ||\boldsymbol{x} - \boldsymbol{y}||$.

Angles between lines.

Path $\gamma : [0, 1] \to \mathbb{E}^N$ has length $L(\gamma) = \int_0^1 ||\gamma'(t)||\, dt$.

Geodesics are segments of straight lines.

1.2 Euclidean Isometries

$T : \mathbb{E}^N \to \mathbb{E}^N$ is an *isometry* if

$$d(T(\boldsymbol{x}), T(\boldsymbol{y})) = d(\boldsymbol{x}, \boldsymbol{y}) \qquad \text{for all } \boldsymbol{x}, \boldsymbol{y} \in \mathbb{E}^N \ .$$

Examples.

Translations: $\boldsymbol{x} \mapsto \boldsymbol{x} + \boldsymbol{t}$.

Orthogonal maps $R \in \mathrm{O}(N)$: $\boldsymbol{x} \mapsto R\boldsymbol{x}$.

$\boldsymbol{x} \mapsto R\boldsymbol{x} + \boldsymbol{t}$.

The isometries form a group: $\mathrm{Isom}(\mathbb{E}^N)$.

Proposition 1.1 Isometries of $\mathbb{E}^N$

Every isometry of the Euclidean N-space $\mathbb{E}^N$ is of the form

$$\boldsymbol{x} \mapsto R\boldsymbol{x} + \boldsymbol{t}$$

for some $R \in \mathrm{O}(N)$ and $\boldsymbol{t} \in \mathbb{R}^N$. Moreover, every such map is an isometry of $\mathbb{E}^N$.

Isometries preserve straight lines and angles.

Orientation of isometries: $T : \boldsymbol{x} \mapsto R\boldsymbol{x} + \boldsymbol{t}$ is
orientation preserving if $\det R = +1$ and
orientation reversing if $\det R = -1$.

Proposition 1.2 Isometries of $\mathbb{E}^2$

An orientation preserving isometry of $\mathbb{E}^2$ *is:*
(a) *The identity.*
(b) *A translation.*
(c) *A rotation about some point* $\boldsymbol{c} \in \mathbb{E}^2$.
An orientation reversing isometry of $\mathbb{E}^2$ *is:*
(d) *A reflection.*
(e) *A glide reflection, that is a reflection in a line* ℓ *followed by a translation parallel to* ℓ.

Lemma 1.3 Orthogonal linear maps in $\mathbb{R}^3$

Let $R : \mathbb{R}^3 \to \mathbb{R}^3$ be an orthogonal linear map. If R is orientation preserving, then either R is the identity or else a rotation about a line m through the origin. If R is orientation reversing, then R is either a reflection in a plane through the origin or else a rotation about an line through the origin followed by reflection in the plane through the origin perpendicular to that line.

Proposition 1.4 Isometries of $\mathbb{E}^3$

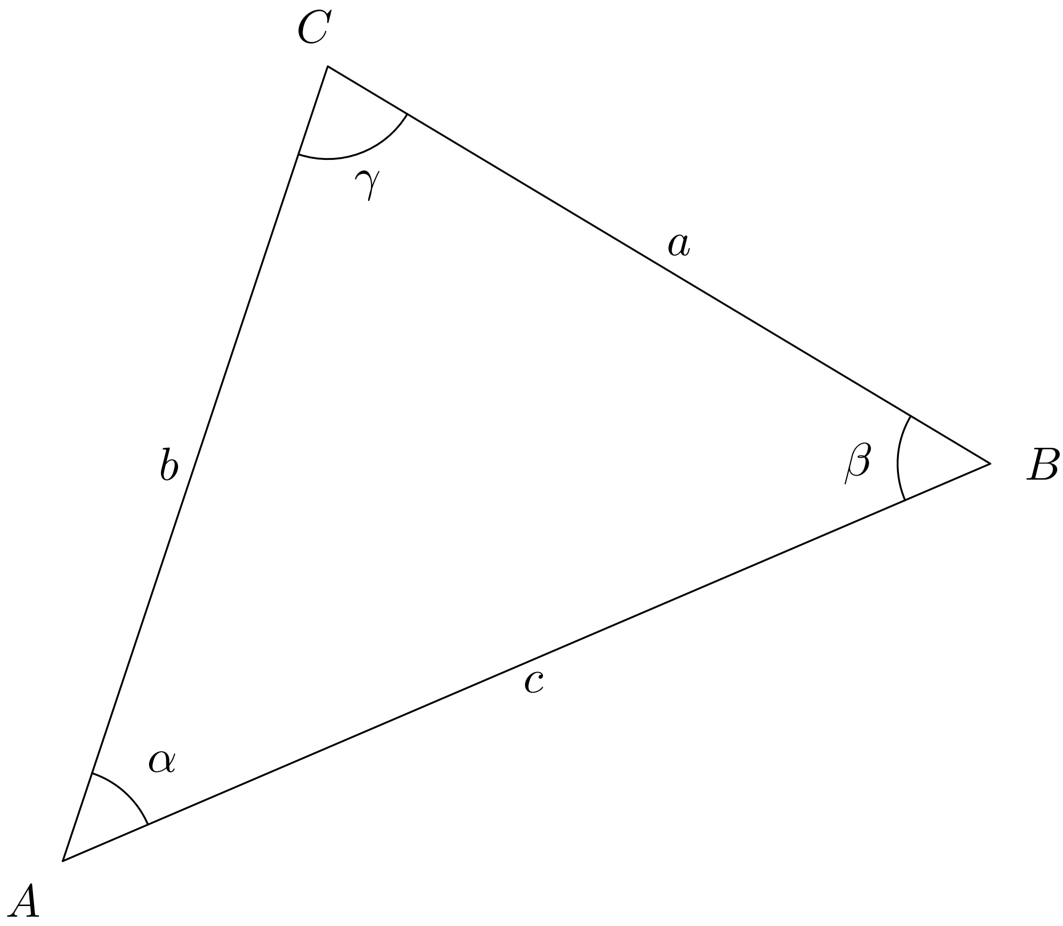
An orientation preserving isometry of Euclidean 3*-space $\mathbb{E}^3$ is:*

(a) *The identity.*
(b) *A translation.*
(c) *A rotation about some line ℓ.*
(d) *A screw rotation, that is a rotation about some line ℓ followed by a translation parallel to ℓ.*

An orientation reversing isometry of $\mathbb{E}^3$ is:

(e) *A reflection in some plane Π.*
(f) *A glide reflection, that is a reflection in a plane Π followed by a translation parallel to Π.*
(g) *A rotatory reflection, that is a rotation about some axis ℓ followed by reflection in a plane perpendicular to ℓ.*

1.3 Euclidean Triangles



We say that two triangles are *isometric* or *congruent* if there is an isometry $T : \mathbb{E}^2 \to \mathbb{E}^2$ that maps one onto the other.

Proposition 1.5 Side lengths determine an Euclidean triangle up to isometry

Two triangles Δ, Δ' *in the Euclidean plane* $\mathbb{E}^2$ *are isometric if and only if they have the same side lengths.*

Two isometric triangles also have the same angles but the converse fails. Two triangles Δ and Δ' that have the same angles are *similar*, that is, there is an enlargement of Δ that is isometric to Δ'.

Proposition 1.6 Sum of angles of an Euclidean triangle

The sum of the angles of an Euclidean triangle is π.

Proposition 1.7 Euclidean Cosine rule

For an Euclidean triangle Δ

$$a^2 = b^2 + c^2 - 2bc\cos\alpha \ .$$

Note that we get other forms of the cosine rule by permuting the vertices of Δ. The case where Δ has a right-angle at $\boldsymbol{A}$ gives Pythagoras' theorem: $a^2 = b^2 + c^2$.

Proposition 1.8 The Euclidean Sine rule

For an Euclidean triangle Δ

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} \ .$$