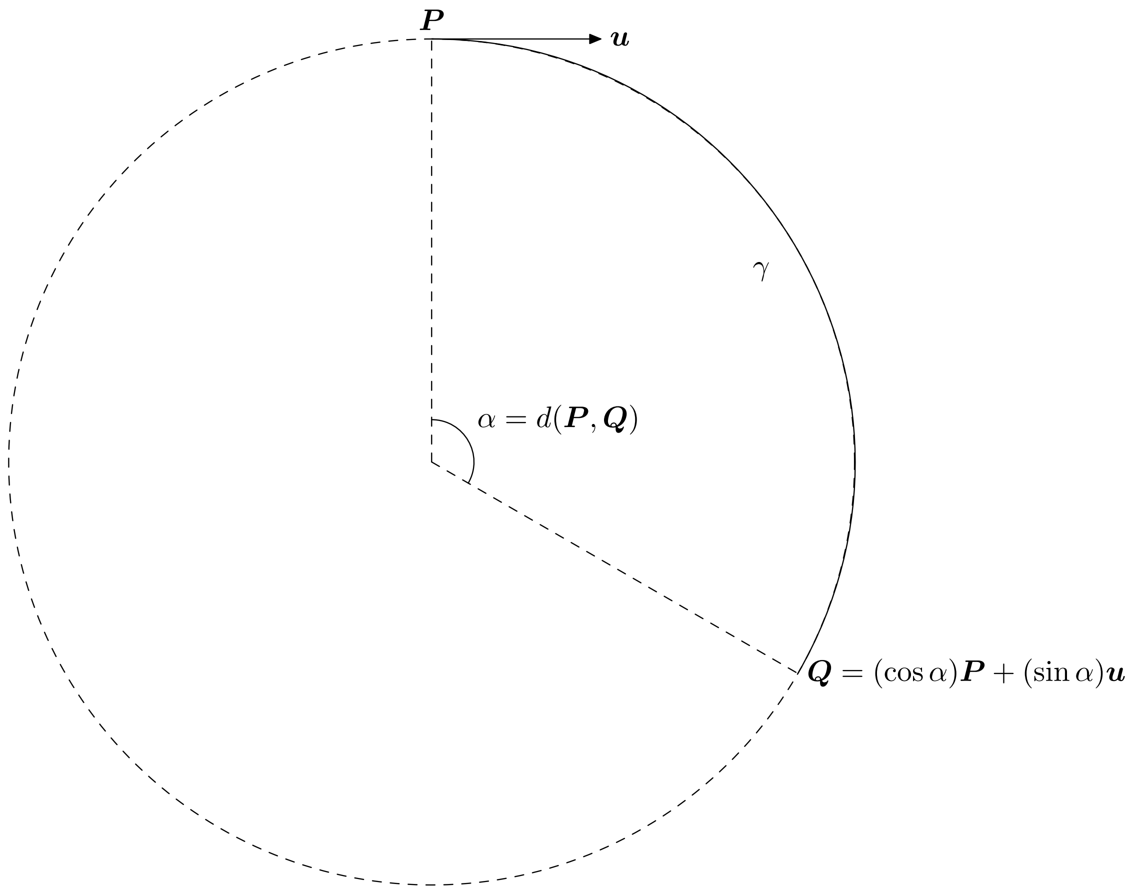
2. THE SPHERE

2.1 The geometry of the sphere

Let $S = S^2 = \{\boldsymbol{x} \in \mathbb{R}^3 : ||\boldsymbol{x}|| = 1\}$ be the unit sphere.

A plane through the origin cuts S in a *great circle*. We call these *spherical lines*.



Proposition 2.1 Geodesics on the sphere

For any two points $\boldsymbol{P}, \boldsymbol{Q} \in S^2$, the shortest path from $\boldsymbol{P}$ to $\boldsymbol{Q}$ follows the shorter arc of a spherical line through $\boldsymbol{P}$ and $\boldsymbol{Q}$. The length of this path is

$$d(\boldsymbol{P}, \boldsymbol{Q}) = \cos^{-1} \boldsymbol{P} \cdot \boldsymbol{Q} \ .$$

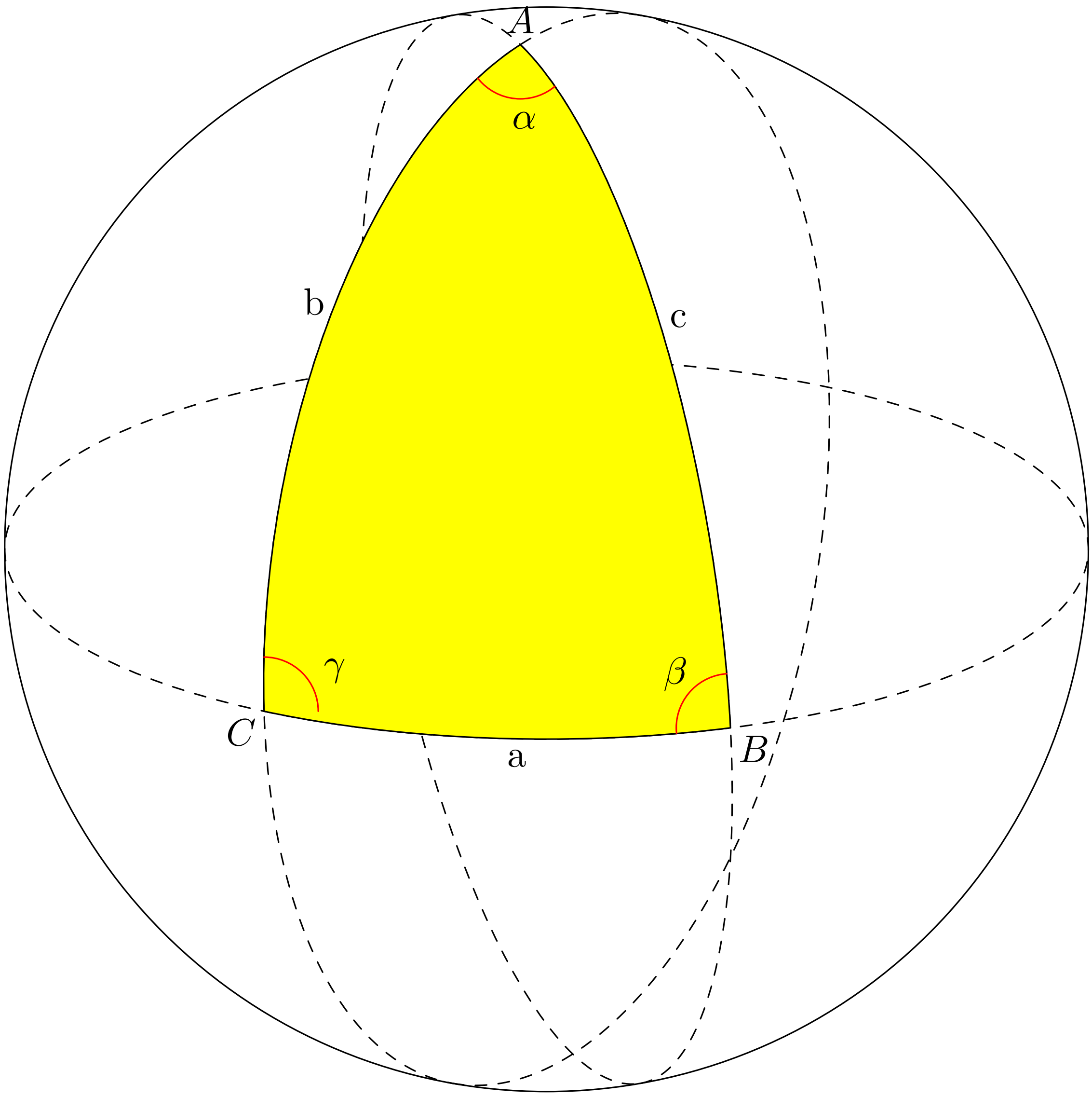
This gives a metric $d(\cdot, \cdot)$ on the sphere.

2.2 Spherical isometries

Proposition 2.2 Isometries of S^2

Every isometry of S^2 *is of the form* $\boldsymbol{x} \mapsto R(\boldsymbol{x})$ *for an orthogonal linear map* $R : \mathbb{R}^3 \to \mathbb{R}^3$.

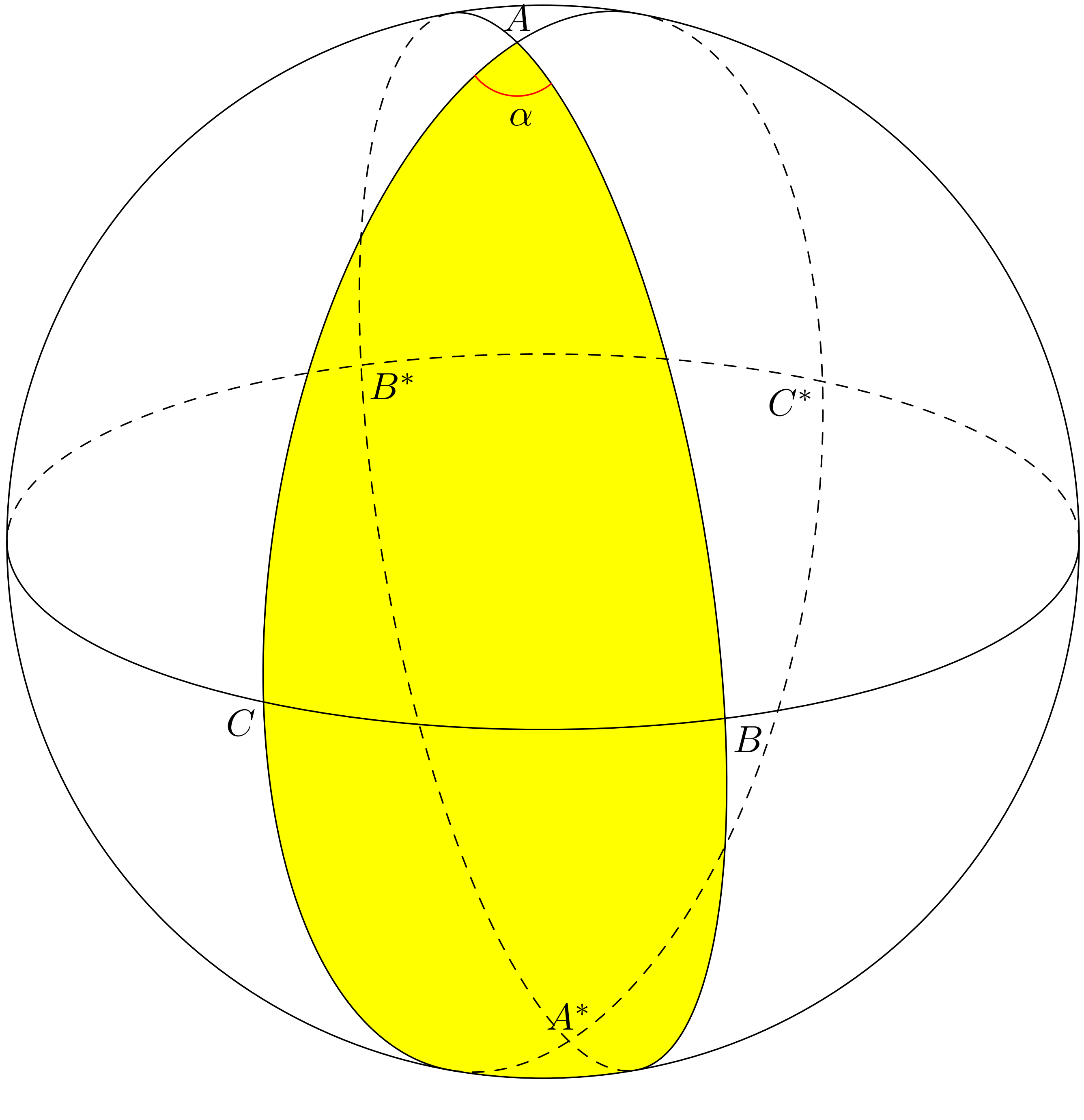
2.3 Spherical Triangles



Proposition 2.3 Gauss – Bonnet theorem for spherical triangles

For a triangle Δ *on the unit sphere* S^2 *with area* $\mathbb{A}(\Delta)$ *we have*

$$\alpha + \beta + \gamma = \pi + \mathbb{A}(\Delta) \ .$$


A
α
B*
C*
C
B
A*

Suppose that we have a polygon P with N sides each of which is an arc of a spherical line. (We will only consider the case where N is at least 1 and the sides of the polygon do not cross one another, so P is simply connected.) If the internal angles of the polygon are $\theta_1, \theta_2, \ldots, \theta_N$, then we can divide it into $N-2$ triangles and obtain

$$\theta_1 + \theta_2 + \ldots + \theta_N = (N-2)\pi + \mathbb{A}(P) \ .$$

Now consider dividing the entire sphere into a finite number of polygonal faces by drawing arcs of spherical lines on the sphere. Let the number of polygonal faces be F, the number of arcs of spherical lines (edges) be E, and the number of vertices of the polygons V. The *Euler number* for this subdivision is $F - E + V$.

Proposition 2.4 Euler's formula for the sphere

Let the sphere be divided into F simply connected faces by drawing E arcs of spherical lines joining V vertices on S^2. Then

$$F - E + V = 2 \ .$$

Proposition 2.5 Spherical Cosine Rule I

For a spherical triangle Δ

$$\cos a = \cos b \cos c + \sin b \sin c \cos \alpha \ .$$

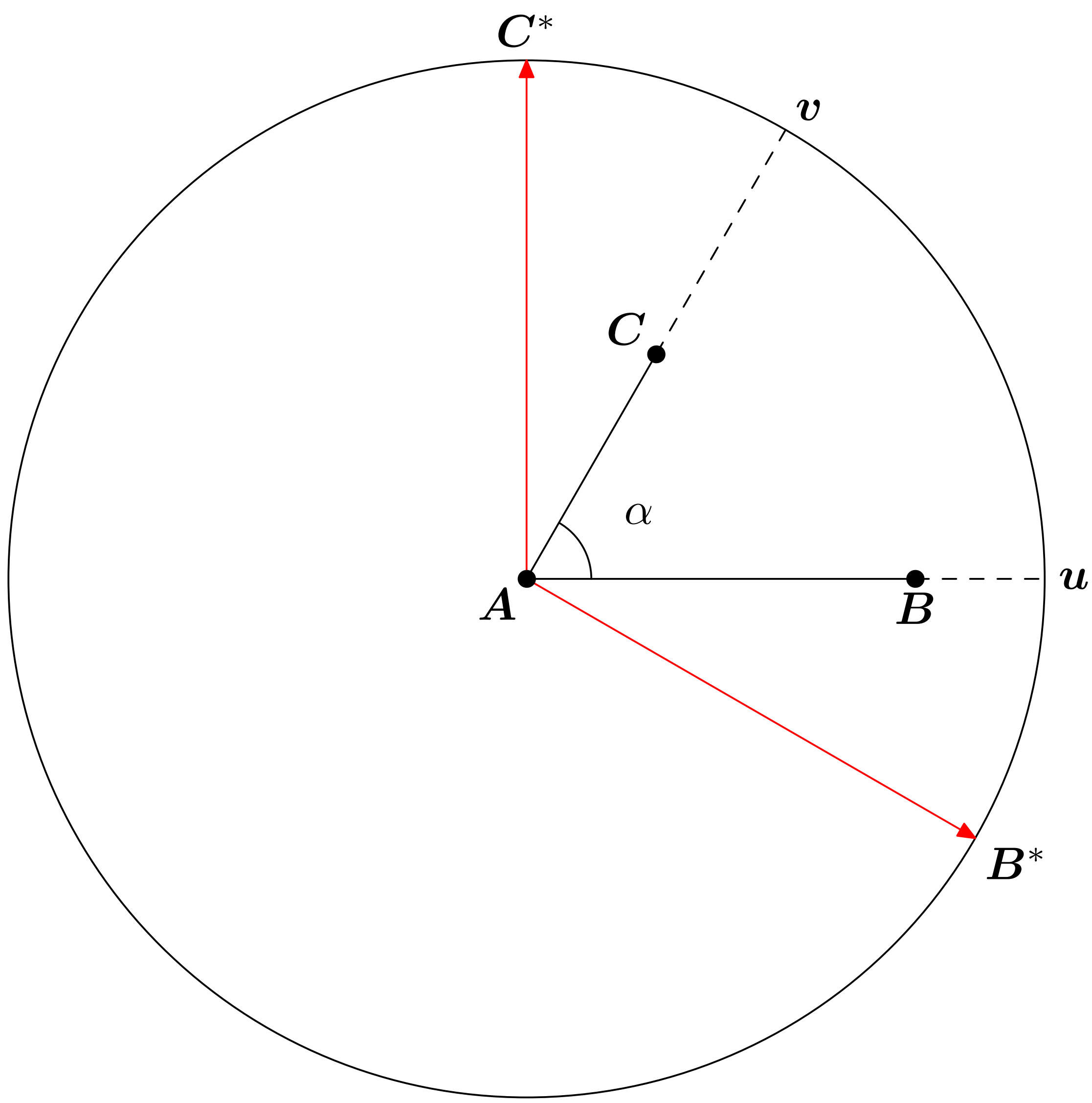
Proposition 2.6 The Spherical Sine rule

For a spherical triangle Δ

$$\frac{\sin a}{\sin \alpha} = \frac{\sin b}{\sin \beta} = \frac{\sin c}{\sin \gamma} \ .$$

Proposition 2.7 Dual spherical triangles

Let Δ be a spherical triangle with angles α, β, γ and side lengths a, b, c. Then the dual triangle Δ^* has sides of length $a^* = \pi - \alpha, b^* = \pi - \beta, c^* = \pi - \gamma$ and angles $\alpha^* = \pi - a, \beta^* = \pi - b, \gamma^* = \pi - c$.



Corollary 2.8 Spherical Cosine Rule II

For a spherical triangle Δ

$$\cos \alpha = -\cos \beta \cos \gamma + \sin \beta \sin \gamma \cos a \ .$$

2.4 The Projective Plane