GEOMETRY AND GROUPS

These notes are to remind you of the results from earlier courses that we will need at some point in this course. The exercises are entirely optional, although they will all be useful later in the course. Asterisks indicate that they are harder.

0.1 Metric Spaces (Metric and Topological Spaces)

A *metric* on a set X is a map $d : X \times X \to [0, \infty)$ that satisfies:

(a) $d(x, y) \geqslant 0$ with equality if and only if $x = y$;

(b) Symmetry: $d(x, y) = d(y, x)$ for all $x, y \in X$;

(c) Triangle Rule: $d(x, y) + d(y, z) \geqslant d(x, z)$ for all $x, y, z \in X$.

A set X with a metric d is called a *metric space*.

For example, the *Euclidean metric* on $\mathbb{R}^N$ is given by $d(\boldsymbol{x}, \boldsymbol{y}) = ||\boldsymbol{x} - \boldsymbol{y}||$ where

$$||\boldsymbol{a}|| = \sqrt{\left(\sum_{n=1}^{N} |a_n|^2\right)}$$

is the norm of a vector $\boldsymbol{a}$. This metric makes $\mathbb{R}^N$ into a metric space and any subset of it is also a metric space.

A *sequence* in X is a map $\mathbb{N} \to X; n \mapsto x_n$. We often denote this sequence by (x_n). This sequence *converges to a limit* $\ell \in X$ when

$$d(x_n, \ell) \to 0 \qquad \text{as } n \to \infty .$$

A subsequence of the sequence (x_n) is given by taking only some of the terms in the sequence. So, a *subsequence* of the sequence (x_n) is given by

$$n \mapsto x_{k(n)}$$

where $k : \mathbb{N} \to \mathbb{N}$ is a strictly increasing function.

A metric space X is *(sequentially) compact* if every sequence from X has a subsequence that converges to a point of X. Recall the very important result:

Theorem 0.1 Compactness in $\mathbb{R}^N$

A subset X of $\mathbb{R}^N$ is (sequentially) compact if and only if it is closed and bounded in $\mathbb{R}^N$.

This shows that the closed ball $\overline{B} = \{\boldsymbol{x} \in \mathbb{R}^N : ||\boldsymbol{x}|| \leqslant 1\}$ is compact but the open ball $B = \{\boldsymbol{x} \in \mathbb{R}^N : ||\boldsymbol{x}|| < 1\}$ is not.

Exercise:

1. A subset S of a metric space X is *discrete* if, for each $a \in S$ there is a $\delta(a) > 0$ with no elements of S lying in $\{x \in X : 0 < d(x, a) < \delta(a)\}$. Show that every discrete subset of a compact metric space is finite.

2. Let $\mathbb{Z}_2$ be the additive group of integers modulo 2. Let C be the set of all functions $f : \mathbb{N} \to \mathbb{Z}_2$, that is all sequences in $\mathbb{Z}_2$. Show that this is an Abelian group with the group operation given by:

$$f + g : n \mapsto f(n) + g(n) \ .$$

Show that:

$$d(f,g) = \begin{cases} 0 & \text{when } f = g; \\ 2^{-k} & \text{when } f(k) \neq g(k) \text{ but } f(j) = g(j) \text{ for } j < k. \end{cases}$$

is a metric on C.

* Show that the map

$$\phi : C \to \mathbb{R} \ ; \quad f \mapsto \sum_{n=1}^{\infty} 2f(n)3^{-n}$$

is a bijection onto a subset K of $[0,1]$. Show that ϕ is a homeomorphism from C onto K. Deduce that C is compact. (A harder problem is to prove this directly.) (C is the Cartesian product of countably many copies of $\mathbb{Z}_2$. K is the Cantor set.)*

0.2 Matrix Groups (Algebra and Geometry, Linear Algebra, Quadratic Mathematics)

Let $M(N, \mathbb{R})$ be the set of $N \times N$ matrices with real entries. A matrix $A \in M(N, \mathbb{R})$ gives a linear map

$$\alpha : \mathbb{R}^N \to \mathbb{R}^N \ ; \ \ \boldsymbol{x} \mapsto A\boldsymbol{x}$$

by multiplying column vectors in $\mathbb{R}^N$ by the matrix A. We will often identify the matrix A with the linear map α. The *transpose* of a matrix $A = (a_{ij}) \in M(N, \mathbb{R})$ is the matrix B with entries $b_{ij} = a_{ji}$. We will denote this by A^t. The *trace* of the matrix A is the sum

$$\operatorname{tr}(A) = \sum_{n=1}^{N} a_{nn}$$

of the diagonal entries in A.

The space $M(N, \mathbb{R})$ of matrices is a vector space of dimension N^2 and we can give it a Euclidean metric. Define the norm of the matrix A to be

$$||A|| = \sqrt{\sum_{m=1}^{N} \sum_{n=1}^{N} |a_{mn}|^2 = \operatorname{tr}(A^t A)} \ .$$

Then $d(A, B) = ||A - B||$ is a metric on $M(N, \mathbb{R})$.

Recall that a matrix A is invertible if and only if the determinant $\det A$ is non-zero. The set of invertible matrices in $M(N, \mathbb{R})$ forms a group under matrix multiplication:

$$\mathrm{GL}(N, \mathbb{R}) = \{A \in M(N, \mathbb{R}) : \det A \neq 0\}$$

called the *general linear group*. The determinant is then a group homomorphism

$$\det : GL(N, \mathbb{R}) \to \mathbb{R} \setminus \{0\}$$

into the multiplicative group of non-zero real numbers. The kernel of this homomorphism is the *special linear group*:

$$\mathrm{SL}(N, \mathbb{R}) = \{A \in M(N, \mathbb{R}) : \det A = 1\} \ .$$

We say that the matrix A, or the linear map α, *preserves orientation* if $\det A > 0$ and *reverses orientation* if $\det A < 0$.

The standard inner product on $\mathbb{R}^N$ is given by

$$\boldsymbol{x} \cdot \boldsymbol{y} = \sum_{n=1}^{N} x_n y_n = \boldsymbol{x}^t \boldsymbol{y} \ .$$

The linear map α is *orthogonal* if it preserves this inner product, so

$$(\alpha(\boldsymbol{x})) \cdot (\alpha(\boldsymbol{y})) = \boldsymbol{x} \cdot \boldsymbol{y} \qquad \text{for all } \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^N \ .$$

This is equivalent to demanding that the matrix A satisfies

$$\boldsymbol{x}^t A^t A \boldsymbol{y} = \boldsymbol{x}^t \boldsymbol{y} \qquad \text{for all } \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^N$$

and this occurs when $A^t A = I$. So the matrix A is *orthogonal* when $A^t A = I$. These matrices form a subgroup of $GL(N, \mathbb{R})$ called the orthogonal group:

$$\mathrm{O}(N) = \{A \in \mathrm{GL}(N, \mathbb{R}) : A^t A = I\} \ .$$

Note that the determinant of an orthogonal matrix A must satisfy $(\det A)^2 = \det A^t \det A = \det I = 1$, so $\det A = \pm 1$. Those with determinant $+1$ form a normal subgroup

$$\mathrm{SO}(N) = \{A \in \mathrm{SL}(N, \mathbb{R}) : A^t A = I\}$$

of $\mathrm{O}(N)$ called the *special orthogonal group.*

Exercise:

3. Show that the matrices in $\mathrm{SO}(2)$ are the rotation matrices:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

for $0 \leqslant \theta < 2\pi$. Show that the matrices in $\mathrm{O}(2) \setminus \mathrm{SO}(2)$ are the reflection matrices:

$$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

corresponding to reflection in the line $y = \tan \frac{1}{2}\theta x$.

Note that each orthogonal matrix $A \in \mathrm{O}(N)$ preserves the norm of vectors:

$$||A\boldsymbol{x}||^2 = (A\boldsymbol{x}) \cdot (A\boldsymbol{x}) = \boldsymbol{x} \cdot \boldsymbol{x} = ||\boldsymbol{x}||^2 \ .$$

So it will preserve the distance between points. In particular an orthogonal matrix will map the *unit sphere*

$$S^{N-1} = \{\boldsymbol{x} \in \mathbb{R}^N : ||\boldsymbol{x}|| = 1\}$$

to itself.

There are very similar results for complex matrices. We will write $M(N, \mathbb{C})$ for the set of $N \times N$ complex matrices. The *general linear group* is

$$\mathrm{GL}(N, \mathbb{C}) = \{A \in M(N, \mathbb{C}) : \det A \neq 0\}$$

and the *special linear group* is the normal subgroup

$$\mathrm{SL}(N, \mathbb{C}) = \{A \in M(N, \mathbb{C}) : \det A = 1\} \ .$$

The standard inner product on $\mathbb{C}^N$ is given by

$$\langle \boldsymbol{w}, \boldsymbol{z} \rangle = \sum_{n=1}^{N} \overline{w}_n z_n \ .$$

A linear map $\alpha : \mathbb{C}^N \to \mathbb{C}^N$ that preserves this inner product is called *unitary.* The corresponding matrix A satisfies $A^* A = I$ where A^* is the conjugate transpose of A having its ij entry equal to $\overline{a_{ji}}$. Therefore the *unitary group* is

$$\mathrm{U}(N) = \{A \in M(N, \mathbb{C}) : A^* A = I\} \ .$$

The determinant of any unitary matrix must have modulus 1. The *special unitary group* is the normal subgroup

$$\mathrm{SU}(N) = \{A \in M(N, \mathbb{C}) : A^* A = I \text{ and } \det A = 1\} \ .$$

Exercise:

4. Show that the orthogonal group $\mathrm{O}(N)$ is a closed and bounded subset of the metric space $M(N, \mathbb{R})$. Deduce that $\mathrm{O}(N)$ is compact.

0.3 The Riemann Sphere. (Algebra and Geometry)

It is useful to add an extra element ∞ to the complex plane to form the extended complex plane $\mathbb{C} \cup \{\infty\}$. It seems that the point at infinity is very different from the other, finite points but Riemann showed that this is no really the case. He did this by representing all of the points of the extended complex plane by points of the unit sphere S^2 in $\mathbb{R}^3$. This sphere is called the *Riemann sphere*.

Let $\mathbb{P}$ be the unit sphere

$$\mathbb{P} = \{(z,t) \in \mathbb{C} \times \mathbb{R} : |z|^2 + t^2 = 1\}$$

in the three dimensional real vector space $\mathbb{C} \times \mathbb{R}$. The "North pole" of this sphere will be denoted by $N = (0,1)$. Stereographic projection maps points of the complex plane to points of the Riemann sphere $\mathbb{P}$ and *vice versa*. Let $z \in \mathbb{C}$. Then the straight line through N and $(z,0)$ crosses the sphere at N and another point $(w,t) \in \mathbb{P}$. We write $\pi(z) = (w,t)$ and define $\pi(\infty) = N$. Then π gives us a map $\pi : \mathbb{C} \cup \{\infty\} \to \mathbb{P}$. This map is invertible, for if (t,w) is any point of $\mathbb{P}$ except N, then the straight line through N and (w,t) will cross the plane $\{(z,s) : s = 0\}$ at a single point $(z,0)$ with $\pi(z) = (w,t)$.

Exercise:

5. Show that stereographic projection maps a complex number $z \in \mathbb{C}$ to the point

$$\left(\frac{2z}{1+|z|^2}, \frac{|z|^2-1}{1+|z|^2}\right)$$

of the Riemann sphere. Find the corresponding formula for the complex number z that corresponds to a point $(w,t) \in \mathbb{P}$.

*6. Let P, P' be two points of the Riemann sphere and z, z' the corresponding points of $\mathbb{C}$. Draw the triangle N, P, P' and show that it is similar to $N, (z',0), (z,0)$. (Note that the order has changed.) Hence deduce that the Euclidean distance from P to P' is given by

$$\frac{2|z-z'|}{\sqrt{1+|z|^2}\sqrt{1+|z'|^2}} \ .$$

This is called the *chordal distance* between z and z'. How should you interpret the formula when one of z or z' is ∞?

0.4 Möbius Transformations (Algebra and Geometry)

Let a, b, c, d be complex numbers with $ad-bc \neq 0$. Then we can define a map $T : \mathbb{C}\cup\{\infty\} \to \mathbb{C}\cup\{\infty\}$ by

$$T : z \mapsto \frac{az+b}{cz+d} \ .$$

Note that $T(\infty) = a/c$ and $T(-d/c) = \infty$. These maps are called *Möbius transformations* and form a group Möb under composition of maps. The map

$$\phi : \mathrm{GL}(2, \mathbb{C}) \to \text{Möb} \ ; \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto T$$

is a group homomorphism.

A matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in the kernel of this homomorphism when

$$\frac{az+b}{cz+d} = z \qquad \text{for all } z \in \mathbb{C} \cup \{\infty\} \ .$$

This occurs if and only if $a = d$ and $b = c = 0$, so $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \lambda I$ for some scalar $\lambda \in \mathbb{C} \setminus \{0\}$. This shows that a Möbius transformation is unaltered when we multiply each of the coefficients a, b, c, d by a non-zero scalar λ. Usually we choose the scalar λ so that the determinant $ad - bc$ is 1. Then the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in $\mathrm{SL}(2, \mathbb{C})$. Now

$$\phi : \mathrm{SL}(2, \mathbb{C}) \to \text{Möb} \ ; \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto T$$

is a group homomorphism whose kernel consists of the two matrices I and $-I$. Consequently, the Möbius group is the quotient $\mathrm{SL}(2, \mathbb{C})/\{I, -I\}$. We denote this quotient by $\mathrm{PSL}(2, \mathbb{C})$ and call it the *projective linear group.*

Exercise:

7. Two complex numbers z, z' are *antipodal* if they correspond under stereographic projection to two points of the Riemann sphere at opposite ends of a diameter. Prove that z and z' are antipodal if and only if $z' = -1/\overline{z}$.

*8. Suppose that the Möbius transformation $T : z \mapsto (az + b)/(cz + d)$ with $ad - bc = 1$ maps any pair of antipodal points to another pair of antipodal points. Show that the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ must lie in the special unitary group.

   Show that any such Möbius transformation preserves the chordal distance between any two points. (Hint: Try to avoid too much algebra. Note that

$$\left\| \begin{pmatrix} z \\ 1 \end{pmatrix} \right\| = \sqrt{1 + |z|^2} \qquad \text{and} \qquad \left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{matrix} z \\ 1 \end{matrix} \right\| = \sqrt{1 + |T(z)|^2} \ . \Bigg) *$$
