GEOMETRY AND GROUPS

TKC Michaelmas 2006

Sample Section I questions

The Faculty asks that questions in Section I of the Part II examinations are straightforward. They should test your knowledge of the course and not involve significant unseen problems to be solved. I consider the questions below to be appropriate. They have been chosen to cover different aspects of the lectures.

However, these sample questions have not been considered and moderated by examiners.

At the end of this paper are copies of the examination questions that were actually set in previous years (2005, 2006).

1. Show that a map $T : \mathbb{R}^2 \to \mathbb{R}^2$ is an isometry for the Euclidean metric on the plane $\mathbb{R}^2$ if and only if there is a vector $\boldsymbol{v} \in \mathbb{R}^2$ and an orthogonal linear map $B \in \mathrm{O}(2)$ with
$$T(\boldsymbol{x}) = B(\boldsymbol{x}) + \boldsymbol{v} \qquad \text{for all } \boldsymbol{x} \in \mathbb{R}^2 \ .$$
When T is an isometry with $\det B = -1$, show that T is either a reflection or a glide reflection.
2. A finite group G consists entirely of orientation preserving isometries of Euclidean 3-space. Show that either there is a straight line ℓ which each element of G maps onto itself, or else G is the group of all orientation preserving symmetries of one of the Platonic solids. (You may assume that the Platonic solids exist and use any properties of them that you wish.)
3. Define a *lattice* in $\mathbb{R}^2$. Show that such a lattice is either $\{\boldsymbol{0}\}$, or $\mathbb{Z}\boldsymbol{w}_1$, or $\mathbb{Z}\boldsymbol{w}_1 + \mathbb{Z}\boldsymbol{w}_2$ for a pair $\boldsymbol{w}_1, \boldsymbol{w}_2$ of linearly independent vectors.
4. Explain what a *frieze pattern* is and prove that there are exactly 7 distinct symmetry group of frieze patterns.
5. What is a 2-dimensional Euclidean crystallographic group? For such a group define the corresponding lattice and point group.

   Prove that a non-trivial rotation in the point group of a 2-dimensional Euclidean crystallographic group must have order $2, 3, 4$ or 6.
6. Prove that a Möbius transformation is an isometry of the Riemann sphere for the chordal metric if and only if it can be represented as
$$z \mapsto \frac{az+b}{cz+d}$$
for some special unitary matrix
$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SU}(2) \ .$$
7. A non-identity Möbius transformation
$$T : z \mapsto \frac{az+b}{cz+d}$$
has $ad - bc = 1$. Show that the trace $a + d$ determines the conjugacy class of T.
8. Define inversion in a circle Γ on the Riemann sphere. Show from your definition that inversion in Γ exists and is unique for each circle Γ.

   Prove that every Möbius transformation of the Riemann sphere is the composition of an even number of inversions.
9. Define the hyperbolic metric on the unit disc $\mathbb{D}$ and describe the geodesics for this metric. Show that inversion in a circle Γ is an isometry for the hyperbolic metric on $\mathbb{D}$ if and only if Γ is a circle orthogonal to the unit circle $\partial\mathbb{D}$.
10. Show that a loxodromic Möbius transformation can never map the unit disc onto itself. Can a loxodromic transformation ever map the unit circle onto itself?

11. Define the modular group acting on the upper half-plane $\mathbb{R}^2_+$. Show that the set

$$\{z = x + iy \in \mathbb{R}^2_+ : -\tfrac{1}{2} < x < \tfrac{1}{2} \ \text{ and } \ |z| > 1\} \ ,$$

together with part of its boundary, is a fundamental set for the modular group.

12. Define hyperbolic 3-space. Explain carefully how a Möbius transformation can be extended to give an isometry of hyperbolic 3-space. Your explanation should include a proof that the extension of each Möbius transformation is unique.
13. Explain briefly how Möbius transformations of the Riemann sphere are extended to give isometries of the unit ball $B^3 \subset \mathbb{R}^3$ for the hyperbolic metric. Show that every orientation preserving isometry of the hyperbolic metric on B^3 is an extension of a Möbius transformation.
14. Let $\mathbb{H}^3$ be the unit ball in $\mathbb{R}^3$ with the hyperbolic metric. A Möbius transformation R is an *involution* if $R^2 = I$. Describe how such an involution acts on $\mathbb{H}^3$. Describe the set of points of $\mathbb{H}^3$ fixed by the involution R.

    Prove that every Möbius transformation can be expressed as the composition of two involutions. Are these two involutions unique?
15. Define a Kleinian group. Prove that every finite Kleinian group is conjugate to a subgroup of the special orthogonal group SO(3).
16. What does it mean for a group G of Möbius transformations to act discontinuously on hyperbolic 3-space $\mathbb{H}^3$? Show that G acts discontinuously on $\mathbb{H}^3$ if and only if G is a discrete group.
17. Define the limit set of a Kleinian group. Prove from your definition that the limit set depends only on the group.
18. Define the Cantor set and prove carefully that its Hausdorff dimension is $\log 2/\log 3$.
19. Define the Schottky group corresponding to K pairs of discs all of which are disjoint. Explain why this Schottky group is a free group.
20. Define Schottky groups. Explain how a Schottky group can be though of as acting on either the Riemann sphere or on hyperbolic 3-space. Identify, up to homeomorphism, the quotient of hyperbolic 3-space by a Schottky group.

Paper 1, Section I, Question 3, 2005

Let G be a subgroup of the group of isometries $\mathrm{Isom}(\mathbb{R}^2)$ of the Euclidean plane. What does it mean to say that G is *discrete*?

Supposing that G is discrete, show that the subgroup G_T of G consisting of all translations in G is generated by translations in at most two linearly independent vectors in $\mathbb{R}^2$. Show that there is a homomorphism $G \to O(2)$ with kernel G_T.

Draw, and briefly explain, pictures which illustrate two different possibilities for G when G_T is isomorphic to the additive group $\mathbb{Z}$.

Paper 1, Section II, Question 12, 2005

(*This is on material not covered in 2006-2007.*)

What is the *limit set* of a subgroup G of Möbius transformations?

Suppose that G is complicated and has no finite orbit in $\mathbb{C} \cup \{\infty\}$. Prove that the limit set of G is infinite. Can the limit set be countable?

State Jørgensen's inequality, and deduce that not every two-generator subgroup $G = \langle A, B\rangle$ of Möbius transformations is discrete. Briefly describe two examples of discrete two-generator subgroups, one for which the limit set is connected and one for which it is disconnected.

Paper 2, Section I, Question 3, 2005

Describe the *geodesics* in the disc model of the hyperbolic plane $\mathbb{H}^2$.

Define the *area* of a region in $\mathbb{H}^2$. Compute the area $A(r)$ of a hyperbolic circle of radius r from the definition just given. Compute the circumference $C(r)$ of a hyperbolic circle of radius r, and check explicitly that $dA(r)/dr = C(r)$.

How could you define π geometrically if you lived in $\mathbb{H}^2$? Briefly justify your answer.

Paper 3, Section I, Question 3, 2005
By considering fixed points in $\mathbb{C} \cup \{\infty\}$, prove that any complex Möbius transformation is conjugate either to a map of the form $z \mapsto kz$ for some $k \in \mathbb{C}$ or to $z \mapsto z+1$. Deduce that two Möbius transformations g,h (neither the identity) are conjugate if and only if $\mathrm{tr}^2(g) = \mathrm{tr}^2(h)$.

Does every Möbius transformation g also have a fixed point in $\mathbb{H}^3$? Briefly justify your answer.

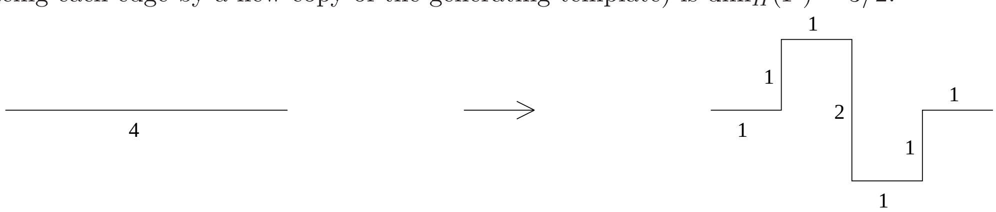
Paper 4, Section I, Question 3, 2005
Show that a set $F \subset \mathbb{R}^n$ with Hausdorff dimension strictly less than one is totally disconnected.

What does it mean for a Möbius transformation to *pair two discs*? By considering a pair of disjoint discs and a pair of tangent discs, or otherwise, explain in words why there is a 2-generator Schottky group with limit set $\Lambda \subset \mathbb{S}^2$ which has Hausdorff dimension at least 1 but which is not homeomorphic to a circle.

Paper 4, Section II, Question 12, 2005
For real $s \geqslant 0$ and $F \subset \mathbb{R}^n$, give a careful definition of the s-dimensional *Hausdorff measure* of F and of the *Hausdorff dimension* $\dim_H(F)$ of F.

For $1 \leqslant i \leqslant k$, suppose $S_i : \mathbb{R}^n \to \mathbb{R}^n$ is a similarity with contraction factor $c_i \in (0,1)$. Prove there is a unique non-empty compact invariant set I for the $\{S_i\}$. State a formula for the Hausdorff dimension of I, under an assumption on the S_i you should state.

Hence show the Hausdorff dimension of the fractal F given by iterating the scheme below (at each stage replacing each edge by a new copy of the generating template) is $\dim_H(F) = 3/2$.



[Numbers denote lengths]

Paper 1, Section I, Question 3, 2006
Suppose $S_i : \mathbb{R}^n \to \mathbb{R}^n$ is a similarity with contraction factor $c_i \in (0,1)$ for $1 \leqslant i \leqslant k$. Let X be the unique non-empty compact invariant set for the S_i's. State a formula for the Hausdorff dimension of X, under an assumption on the S_i's you should state. Hence compute the Hausdorff dimension of the subset X of the square $[0,1]^2$ defined by dividing the square into a 5×5 array of squares, removing the open middle square $(2/5, 3/5)^2$, then removing the middle 1/25th of each of the remaining 24 squares, and so on.

Paper 1, Section II, Question 12, 2006
Compute the area of the ball of radius r around a point in the hyperbolic plane. Deduce that, for any tessellation of the hyperbolic plane by congruent, compact tiles, the number of tiles which are at most n "steps" away from a given tile grows exponentially in n. Give an explicit example of a tessellation of the hyperbolic plane.

Paper 2, Section I, Question 3, 2006
Determine whether the following elements of $\mathrm{PSL}_2(\mathbb{R})$ are elliptic, parabolic, or hyperbolic. Justify your answers.

$$\begin{pmatrix} 5 & 8 \\ -2 & -3 \end{pmatrix}, \qquad \begin{pmatrix} -3 & 1 \\ 2 & -1 \end{pmatrix}.$$

In the case of the first of these transformations find the fixed points.

Paper 3, Section I, Question 3, 2006
Let G be a discrete subgroup of the Möbius group. Define the limit set of G in S^2. If G contains two loxodromic elements whose fixed point sets in S^2 are different, show that the limit set of G contains no isolated points.

Paper 4, Section I, Question 3, 2006
What is a crystallographic group in the Euclidean plane? Prove that, if G is crystallographic and g is a nontrivial rotation in G, then g has order 2, 3, 4, or 6.

Paper 4, Section II, Question 12, 2006
Let G be a discrete subgroup of $\mathrm{PSL}_2(\mathbb{C})$. Show that G is countable. Let $G = \{g_1, g_2, \ldots\}$ be some enumeration of the elements of G. Show that for any point p in hyperbolic 3-space $\mathbb{H}^3$, the distance $d_{hyp}(p, g_n(p))$ tends to infinity. Deduce that a subgroup G of $\mathrm{PSL}_2(\mathbb{C})$ is discrete if and only if it acts properly discontinuously on $\mathbb{H}^3$.