11 COMPLEX ANALYSIS IN $\mathbb{C}$

1.1 Holomorphic Functions

A *domain* Ω in the complex plane $\mathbb{C}$ is a connected, open subset of $\mathbb{C}$. Let $z_o \in \Omega$ and f a map $f : \Omega \to \mathbb{C}$. We say that f is *real differentiable* at z_o if there is a *real* linear map $T : \mathbb{C} \to \mathbb{C}$ with

$$f(z_o + w) = f(z_o) + Tw + o(w) \qquad \text{as } w \to 0.$$

T is the *derivative* of f at z_o which we denote by $f'(z_o)$. This real linear map can be expressed as

$$T : w \mapsto \lambda w + \mu \overline{w}$$

for some complex numbers λ and μ. We shall write $\frac{\partial f}{\partial z}(z_o)$ for λ and $\frac{\partial f}{\partial \overline{z}}(z_o)$ for μ. Note that these are not actually partial derivatives although they do share many of the formal properties of partial derivatives. We have

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \quad ; \quad \frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

If the map $T = Df(z_o)$ is *complex* linear then we say that f is *complex differentiable* at z_o. The map T will then be multiplication by a complex number which we call $f'(z_o)$. Hence, a real differentiable function f is complex differentiable at z_o if, and only if, $\frac{\partial f}{\partial \overline{z}}(z_o) = 0$ and then $\frac{\partial f}{\partial z}(z_o) = f'(z_o)$. These are the Cauchy- Riemann equations.

A map $f : \Omega \to \mathbb{C}$ is *holomorphic* if it is complex differentiable at each point of the domain Ω. The collection of all such analytic maps from a domain Ω into $\mathbb{C}$ forms a vector space $\mathcal{O}(\Omega)$. A map $g : \Omega \to \Omega'$ between two domains is *conformal* if it is analytic and has an analytic inverse. When such a map g exists we say that the domains Ω and Ω' are *conformally equivalent*.

Proposition 1.1.1

Let $f : D \to \mathbb{C}$ be an analytic function and $z_o \in D$. Either f is constant or there is a natural number $N \in \{1, 2, 3, \ldots\}$ and a conformal map $g : D \to D'$ from a neighbourhood U of z_o in D to a neighbourhood V of 0 with

$$f(z) = f(z_o) + g(z)^N \qquad \text{for } z \in U.$$

The number N is unique and is called the *degree* $\deg f(z_o)$ of f at z_o.

Proof:

Suppose that f is not constant. Then there must be a least N with $f^{(N)}(z_o) \neq 0$. The Taylor expansion for f shows that

$$f(z) = f(z_o) + (z - z_o)^N h(z) \qquad \text{for } z \in \Omega$$

for some analytic function $h : D \to \mathbb{C}$ with $h(z_o) \neq 0$. Since h is continuous there is a disc U_o about z_o with $\Re(h(z)/h(z_o)) > 0$ for $z \in U_o$. So h has an analytic Nth root $k : \Omega \to \mathbb{C}$. Then

$$f(z) = f(z_o) + ((z - z_o)k(z))^N$$

so we can set $g(z) = (z - z_o)k(z)$. Now $g(z_o) = 0$ and $g'(z_o) = k(z_o) \neq 0$ so the inverse function theorem shows that there is a neighbourhood U of z_o, contained in U_o, with $g : U \to V$ conformal. □

The *critical* points of a non-constant analytic function $f : D \to \mathbb{C}$ are those z_o where $f'(z_o) = 0$. Because the zeros of f' are isolated, these form a discrete, and hence countable, subset of D. Note that $f'(z_o) = 0$ if, and only if, $\deg f(z_o) > 1$.

We say that $f : D \to \mathbb{C}$ is *locally conformal* if, for each point $z_o \in D$, there are open, connected neighbourhoods U of z_o in D and V of $f(z_o)$ with $f|_U : U \to V$ conformal. The previous proposition shows that f is locally conformal if and only if $\deg f(z_o) = 1$ for every point $z_o \in D$.

Proposition 1.1.2

Let $f : D \to E$, $g : E \to \mathbb{C}$ be holomorphic functions. Then $g \circ f : D \to \mathbb{C}$ is holomorphic and

$$\deg(g \circ f)(z_o) = \deg f(z_o).\deg g(f(z_o))$$

for each $z_o \in D$.

Proof:

If we set $M = \deg f(z_o)$ and $N = \deg g(f(z_o))$, then

$$f(z) - f(z_o) = (z - z_o)^M \phi(z) \quad \text{and} \quad g(w) - g(f(z_o)) = (w - f(z_o))^N \gamma(w)$$

for holomorphic functions ϕ, γ with $\phi(z_o) \neq 0$ and $\gamma(f(z_o)) \neq 0$. Therefore

$$g(f(z)) - g(f(z_o)) = (f(z) - f(z_o))^N \gamma(f(z)) = ((z - z_o)^M \phi(z))^N \gamma(f(z)) = (z - z_o)^{MN} \left(\phi(z)^N \gamma(f(z))\right) ,$$

with $\phi(z_o)^N \gamma(f(z_o)) \neq 0$. □

1.2 Locally Uniform Convergence

Let $f_n, f : \Omega \to \mathbb{C}$ be functions on a domain Ω. We say that f_n converges to f *locally uniformly* on Ω if, for each $z_o \in \Omega$ there is a neighbourhood U of z_o in Ω with $f_n \to f$ uniformly on U. Each compact subset K of Ω is covered by finitely many such neighbourhoods so this will imply that $f_n \to f$ uniformly on K. Also, every neighbourhood in $\mathbb{C}$ contains a compact neighbourhood. Hence $f_n \to f$ locally uniformly on Ω if, and only if, $f_n \to f$ uniformly on each compact subset of Ω.

An increasing sequence (K_n) of compact sets with union Ω is called a *compact exhaustion* of Ω. An example is

$$K_n = \{z \in \Omega : |z| \leqslant n \text{ and } |z - w| \geqslant \tfrac{1}{n} \text{ for each } w \in \mathbb{C} \setminus \Omega\}.$$

The functions f_n converge locally uniformly to f on Ω if, and only if, they converge uniformly on each of the sets in a compact exhaustion of Ω. (* A Riemann surface also has a compact exhaustion but it is very much harder to exhibit one. We will do so in the last chapter when we have proved the Riemann mapping theorem. *)

The topology of locally uniform convergence is a metric topology:

Proposition 1.2.1

Let D be a domain in $\mathbb{C}$. Then there is a topology on $C(D)$ with $f_n \to f$ for this metric if and only if $f_n \to f$ locally uniformly on D.

Proof:

Let (K_n) be a compact exhaustion of D and set

$$d(f, g) = \sum 2^{-n} \min\left(1, ||f - g||_{K_n}\right)$$

where

$$||h||_K = \sup\{|h(z)| : z \in K\} \ .$$

□

Locally uniform convergence is the "correct" type of convergence for analytic functions. Firstly, it arises frequently in complex analysis. For example, the partial sums of a power series converge locally uniformly on the open disc where the power series converges. Secondly we have:

Proposition 1.2.2

Let Ω be a domain in $\mathbb{C}$ and $f_n : \Omega \to \mathbb{C}$ a sequence of analytic functions which converge locally uniformly to $f : \Omega \to \mathbb{C}$. Then f is also analytic. Furthermore, the derivatives f_n' converge locally uniformly to f'.

Proof:

For $z_o \in \Omega$ we can find a disc $D = \{z : |z - z_o| \leqslant r\} \subset \Omega$ with $f_n \to f$ uniformly on D. Then $\int_\gamma f_n \, dz = 0$ for any simple closed curve γ in D because of Cauchy's theorem. The uniform convergence on γ implies that $\int_\gamma f \, dz = 0$. Hence Morera's theorem implies that f is analytic on D.

Cauchy's representation theorem shows that

$$f_n'(w) = \frac{1}{2\pi i} \int_\Gamma \frac{f_n(z)}{(z - w)^2} \, dz$$

for $|w - z_o| < r$ and Γ the circle $\Gamma : t \mapsto z_o + re^{it}$. Since $f_n \to f$ uniformly on Γ we see that $f_n' \to f'$ uniformly on $\{w : |w - z_o| \leqslant \frac{1}{2}r\}$. □

Recall the Arzela-Ascoli theorem:

Let K be compact and $C(K)$ the Banach space of continuous functions $f : K \to \mathbb{C}$ with the uniform norm $||f||_\infty = \sup(|f(z)| : z \in K)$. Then a subset $\mathcal{F}$ of $C(K)$ is relatively compact (i.e. its closure is compact) if, and only if,

(a) $\mathcal{F}$ is bounded : there exists c with $||f||_\infty < c$ for all $f \in \mathcal{F}$.

(b) $\mathcal{F}$ is equicontinuous : for each $z_o \in K$ and $\varepsilon > 0$ there is a neighbourhood U of z_o with

$$|f(z) - f(z_o)| < \varepsilon \qquad \text{for all } f \in \mathcal{F} \quad \text{and all } z \in U.$$

We can use this to prove a similar characterization for relatively compact sets of analytic functions. For a domain Ω we may give the vector space $\mathcal{O}(\Omega)$ a topology – the topology of locally uniform convergence – by taking the sets

$$\{g \in \mathcal{O}(\Omega) : |g(z) - f(z)| < \varepsilon \text{ for all } z \in K\}$$

with K a compact subset of Ω and $\varepsilon > 0$, as a base for the neighbourhoods of f. Then $f_n \to f$ in this topology if, and only if, $f_n \to f$ locally uniformly on Ω. A subset $\mathcal{F}$ of $\mathcal{O}(\Omega)$ is called a *normal family* if, for every compact subset K of Ω there is a constant c_K with

$$|f(z)| \leqslant c_K \qquad \text{for all } f \in \mathcal{F} \quad \text{and all } z \in K.$$

Theorem 1.2.3

A subset $\mathcal{F}$ of $\mathcal{O}(\Omega)$ is relatively compact for the topology of locally uniform convergence if, and only if, $\mathcal{F}$ is a normal family.

Proof:

Suppose that $\mathcal{F}$ is relatively compact. For any compact set $K \subset \Omega$ the restriction map

$$R_K : \mathcal{O}(\Omega) \to C(K) \quad ; \quad f \mapsto f|_K$$

is continuous, so $R_K(\mathcal{F})$ is relatively compact. It must then certainly be bounded. So $\mathcal{F}$ is a normal family.

Conversely, suppose that $\mathcal{F}$ is a normal family. Let $K \subset \Omega$ be compact and consider $R_K(\mathcal{F}) \subset C(K)$. This is certainly bounded. For $z_o \in K$ find a closed disc $D = \{z \in \mathbb{C} : |z - z_o| \leqslant r\}$ contained in Ω. Since D is compact there is a constant c_D which bounds each $f \in \mathcal{F}$ on D. By Cauchy's representation theorem

$$f(w) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - w} \, dz$$

for $|w - z_o| < r$ and γ the circle $\gamma : t \mapsto z_o + re^{it}$. Hence, if $|w - z_o| \leqslant \frac{1}{2}r$, we have

$$\begin{aligned} |f(w) - f(z_o)| &= \left| \frac{1}{2\pi i} \int_\gamma f(z) \left(\frac{1}{z-w} - \frac{1}{z - z_o} \right) \, dz \right| \\ &\leqslant \frac{1}{2\pi} \int_\gamma |f(z)| \left| \frac{w - z_o}{(z-w)(z-z_o)} \right| \, |dz| \\ &\leqslant \frac{1}{2\pi} c_D \frac{|w - z_o|}{\frac{1}{2} r.r} 2\pi r \quad = \quad \frac{2c_D|w - z_o|}{r} \end{aligned}$$

So $R_K(\mathcal{F})$ is equicontinuous and hence relatively compact by the Arzela-Ascoli theorem.

Finally observe that the definition of the topology of locally uniform convergence implies that the mapping

$$\mathcal{O}(\Omega) \to \prod_{K \subset \Omega} C(K) \quad ; \quad f \mapsto (f|_K)$$

is a homeomorphic embedding of $\mathcal{O}(\Omega)$ into the product of $C(K)$ over all compact subsets K of Ω. This maps $\mathcal{F}$ into the product of the sets $R_K(\mathcal{F})$, which have just shown to be compact. By Tychonoff's theorem, $\mathcal{F}$ is relatively compact. □

1.3 Harmonic Functions

A function $u : D \to \mathbb{C}$ on a domain D in $\mathbb{C}$ is *harmonic* if it is twice continuously differentiable and

$$\triangle u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 4 \frac{\partial^2 u}{\partial \overline{z} \partial z} = 0$$

on D. Hence, u is harmonic if, and only if, $\partial u / \partial z : D \to \mathbb{C}$ is holomorphic. It certainly follows that any harmonic function is infinitely differentiable. Furthermore, if $f : \Omega' \to \Omega$ is holomorphic and $u : \Omega \to \mathbb{C}$ is harmonic, then $u \circ f : \Omega' \to \mathbb{C}$ is harmonic.

It is clear that any holomorphic function f is harmonic, as is its conjugate.

Proposition 1.3.1

Every harmonic function on a disc can be expressed as $f + \overline{g}$ for two holomorphic functions f, g on the disc.

Proof:

We may assume that the disc is the unit disc $\mathbb{D}$ and that $u : \mathbb{D} \to \mathbb{C}$ is harmonic. Then $\partial u / \partial z$ is holomorphic on $\mathbb{D}$ and so can be written as a power series:

$$\frac{\partial u}{\partial z} = \sum b_n z^n \qquad \text{for} \ \ z \in \mathbb{D} \ .$$

The conjugate $\overline{u}$ is also harmonic and so

$$\frac{\partial u}{\partial \overline{z}} = \overline{\left(\frac{\partial \overline{u}}{\partial z}\right)} = \overline{\sum c_n z^n} \ .$$

Let $\gamma : [0, 1] \to \mathbb{D}$ be a smooth curve from 0 to a point $z_1 \in \mathbb{D}$. Then

$$u(z_1) - u(0) = \int_0^1 \frac{d}{dt} u(\gamma(t)) \, dt = \int_\gamma \frac{\partial u}{\partial z} \, dz + \int_\gamma \frac{\partial u}{\partial \overline{z}} \, d\overline{z} = \sum \frac{b_n}{n+1} z^{n+1} \ + \ \overline{\sum \frac{c_n}{n+1} z^{n+1}} \ .$$

Thus there are two holomorphic functions f, g on $\mathbb{D}$ with $u = f + \overline{g}$. □

Note that it is not true that every harmonic function on an arbitrary domain Ω is the sum $f + \overline{g}$ for two holomorphic functions on Ω. For example, the function $\log |z|$ is harmonic on $\mathbb{C} \setminus \{0\}$ but can not be expressed in this way.

When $u : \Omega \to \mathbb{R}$ is a real-valued harmonic function, then the Proposition shows that u is the real part of a holomorphic function on any disc in Ω. Again, it need not be the real part of a holomorphic function on all of Ω.

Let $U : \mathbb{D}(z_o, R) \to \mathbb{C}$ be a harmonic function on a disc. Then Proposition 1.3.1 shows that

$$u(z) = \sum_{n=0}^{\infty} a_n (z - z_o)^n + \overline{\sum_{n=-\infty}^{-1} a_n (z - z_o)^n}$$

for some coefficients (a_n). Hence

$$u(z_o + re^{i\theta}) = \sum_{n=-\infty}^{\infty} a_n r^{|n|} e^{in\theta} \ .$$

The series converges uniformly on $\{z : |z - z_o| = r\}$ for any fixed r with $0 \leqslant r < R$. So

$$a_n = r^{-|n|} \int_0^{2\pi} u(z_o + re^{i\theta}) e^{-in\theta} \ \frac{d\theta}{2\pi} \ .$$

In particular,

$$u(z_o) = a_0 = \int_0^{2\pi} u(z_o + re^{i\theta}) \ \frac{d\theta}{2\pi} \ .$$

This is the *mean value property for harmonic functions.*

Proposition 1.3.2 Maximum principle for harmonic functions.

If the harmonic function $u : \Omega \to \mathbb{R}$ on the domain $\Omega \subset \mathbb{C}$ has a local maximum (or a local minimum) then it is constant.

Proof:

Suppose that u has a local maximum at z_o. Then there is a disc Δ containing z_o with $u(z) \leqslant u(z_o)$ for all $z \in \Delta$. We have shown that there is an analytic function $a : \Delta \to \mathbb{C}$ with $u = \Re\, a$. So $\Re\, a(z) \leqslant \Re\, a(z_o)$ and this certainly implies that a is not an open mapping. Hence a must be constant, and u must be constant on Δ.

The zeros of the analytic function $\frac{\partial u}{\partial z}$ are therefore not isolated, so it must be identically 0. Thus u is constant on all of Ω. □

We wish to study the local behaviour of harmonic functions, so we look in detail at harmonic functions on the unit disc. Let

$$\mathcal{H}(D) = \{u : \overline{\mathbb{D}} \to \mathbb{R} : u \text{ is continuous and harmonic on } \mathbb{D}\}.$$

This is clearly a vector space. We will give it the supremum norm $||u||_\infty = \sup(|u(z)| : z \in \overline{\mathbb{D}})$. For $u \in \mathcal{H}(\mathbb{D})$ the restriction $Ru = u|\partial\mathbb{D}$ is in the space $C(\partial\mathbb{D})$ of continuous functions on the unit circle and $R : \mathcal{H}(\mathbb{D}) \to C(\partial\mathbb{D})$ is a continuous linear map. The maximum principle shows that R preserves the norm. Conversely, for $f \in C(\partial\mathbb{D})$ define the *Poisson integral* $Pf : \overline{\mathbb{D}} \to \mathbb{R}$ by $Pf(z) = f(z)$ for $z \in \partial\mathbb{D}$ and

$$Pf(z) = \int_0^{2\pi} f(e^{i\theta}) \frac{1-|z|^2}{|e^{i\theta}-z|^2}\,\frac{d\theta}{2\pi} \qquad \text{for } z \in \mathbb{D}.$$

We will prove that $Pf \in \mathcal{H}(\mathbb{D})$. The expression

$$\frac{1-|z|^2}{|e^{i\theta}-z|^2} \;=\; \Re\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right)$$

is called the *Poisson kernel* for the disc.

Theorem 1.3.2 Poisson's formula.

For each $f \in C(\partial\mathbb{D})$ the Poisson integral Pf is in $\mathcal{H}(\mathbb{D})$.

Proof:

For $z \in \mathbb{D}$ we have

$$Pf(z) = \Re\int_0^{2\pi} f(e^{i\theta})\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right)\frac{d\theta}{2\pi} = \Re\frac{1}{2\pi i}\int_{\partial\mathbb{D}} f(w)\frac{w+z}{(w-z)w}\,dw.$$

The integral certainly gives an analytic function on $\mathbb{D}$ so Pf is harmonic on $\mathbb{D}$. To complete the proof we need to show that Pf is continuous on $\overline{\mathbb{D}}$. Note also that when we take $f \equiv 1$ the above formula gives

$$1 = P1(z) = \int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|^2}\,\frac{d\theta}{2\pi} \qquad (*)$$

For $0 < r < 1$ set $f_r(e^{i\theta}) = Pf(re^{i\theta})$. Then each f_r is continuous on the unit circle. It will suffice to prove that $f_r \to f$ uniformly as $r \nearrow 1$. Equation $(*)$ shows that

$$\begin{aligned}|f(e^{i\phi}) - f_r(e^{i\phi})| &= \left|\int_0^{2\pi} (f(e^{i\phi}) - f(e^{i\theta}))\,\frac{1-r^2}{|e^{i\theta}-re^{i\phi}|^2}\,\frac{d\theta}{2\pi}\right| \\ &\leqslant \int_0^{2\pi} |f(e^{i\phi}) - f(e^{i\theta})|\,\frac{1-r^2}{|e^{i\theta}-re^{i\phi}|^2}\,\frac{d\theta}{2\pi}\end{aligned}$$

Since $\partial\mathbb{D}$ is compact, f is uniformly continuous so, for $\varepsilon > 0$, there exists $\delta > 0$ with $|f(w_1) - f(w_2)| < \varepsilon$ whenever $|w_1 - w_2| < \delta$. If $|w_1 - w_2| \geqslant \delta \quad (w_1, w_2 \in \partial\mathbb{D})$, then

$$\frac{1-r^2}{|w_1 - rw_2|^2} \leqslant \frac{1-r^2}{|(w_1-w_2)+(1-r)w_2|^2} \leqslant \frac{1-r^2}{(\delta-(1-r))^2}$$

for $0 < r < 1$. The right side of this tends to 0 as $r \nearrow 1$ so there exists r_o with

$$\frac{1-r^2}{|w_1 - rw_2|^2} \leqslant \varepsilon$$

whenever $r_o < r < 1$ and $|w_1 - w_2| \geqslant \delta$. Hence, for $r_o < r < 1$ we obtain

$$\begin{aligned}|f(e^{i\phi}) - f_r(e^{i\phi})| &\leqslant \int_{|e^{i\theta}-e^{i\phi}|<\delta} + \int_{|e^{i\theta}-e^{i\phi}|\geqslant\delta} |f(e^{i\phi}) - f(e^{i\theta})|\,\frac{1-r^2}{|e^{i\theta}-re^{i\phi}|^2}\,\frac{d\theta}{2\pi} \\ &\leqslant \int_{|e^{i\theta}-e^{i\phi}|<\delta} \varepsilon\,\frac{1-r^2}{|e^{i\theta}-re^{i\phi}|^2}\,\frac{d\theta}{2\pi} + \int_{|e^{i\theta}-e^{i\phi}|\geqslant\delta} 2||f||_\infty\,\varepsilon\,\frac{d\theta}{2\pi} \\ &\leqslant \varepsilon + 2||f||_\infty\varepsilon\end{aligned}$$

Therefore, $f_r \to f$ uniformly as $r \nearrow 1$. □

Theorem 1.3.3

The maps $R : \mathcal{H}(\mathbb{D}) \to C(\partial\mathbb{D})$ and $P : C(\partial\mathbb{D}) \to \mathcal{H}(\mathbb{D})$ are mutually inverse linear isometries.

Proof:

We have already seen that R is linear and preserves the norm. Also, P is linear with $RP = I$. So R is surjective. Suppose that $Ru_1 = Ru_2$. Then the difference $u = u_1 - u_2 \in \mathcal{H}(\mathbb{D})$ is 0 on $\partial\mathbb{D}$. By the maximum (and minimum) principle, u is 0 on all of $\mathbb{D}$. Thus R is bijective. Since $RP = I$ we see that P must be the inverse of R and P must be an isometry because R is. □

So, for any $f \in C(\partial\mathbb{D})$ there is an unique $u \in \mathcal{H}(\mathbb{D})$ whose restriction to the boundary is f. Moreover u is given by the Poisson integral Pf. Therefore,

$$u(z) = \int_0^{2\pi} u(e^{i\theta})\, \frac{1 - |z|^2}{|e^{i\theta} - z|^2}\, \frac{d\theta}{2\pi} \qquad (\dagger)$$

for $z \in \mathbb{D}$ and any $u \in \mathcal{H}(\mathbb{D})$. A particularly important case is when $z = 0$ when we see that

$$u(0) = \int_0^{2\pi} u(e^{i\theta})\, \frac{d\theta}{2\pi}$$

which is the mean value of u over the unit circle. This shows that any harmonic function u on a domain Ω has the mean value property:

$$u(z) = \int_0^{2\pi} u(z + re^{i\theta})\, \frac{d\theta}{2\pi}$$

whenever the disc $\{w : |w - z| \leqslant r\}$ lies inside the domain Ω.

Corollary 1.3.4

If $v_n : \Omega \to \mathbb{R}$ are harmonic functions on a domain $\Omega \subset \mathbb{C}$ which converge locally uniformly to $v : \Omega \to \mathbb{R}$ then v is also harmonic. Furthermore the derivatives $\frac{\partial v_n}{\partial z}$ converge locally uniformly to $\frac{\partial v}{\partial z}$.

Proof:

The theorem shows that $\mathcal{H}(\mathbb{D})$ is a Banach space isometric to $C(\partial\mathbb{D})$. Hence the uniform limit of functions in $\mathcal{H}(\mathbb{D})$ is also in $\mathcal{H}(\mathbb{D})$. Hence, for any compact disc $\Delta \subset \Omega$ we have the limit v harmonic on Δ.

Similarly any $u \in \mathcal{H}(\mathbb{D})$ satisfies ($\dagger$) so we can differentiate to obtain

$$\frac{\partial u}{\partial z}(z) = \int_0^{2\pi} u(e^{i\theta})\, \frac{e^{i\theta}}{(e^{i\theta} - z)^2}\, \frac{d\theta}{2\pi} = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} u(w) \frac{1}{(w - z)^2}\, dw.$$

It is now apparent that if the functions $u_n \in \mathcal{H}(\mathbb{D})$ converge uniformly to u on $\partial\mathbb{D}$ then $\frac{\partial u_n}{\partial z}$ converges uniformly to $\frac{\partial u}{\partial z}$ on the disc $\{z \in \mathbb{D} : |z| \leqslant \frac{1}{2}\}$. It follows, as above, that the derivatives of v_n will converge locally uniformly to the derivative of v on Ω. □

Theorem 1.3.5 Harnack's inequality : differential form.

For a compact subset K of a domain $\Omega \subset \mathbb{C}$ there is a constant c with

$$\left| \frac{\partial u}{\partial z}(z) \right| \leqslant cu(z) \qquad \text{for } z \in K$$

and for every positive, harmonic function $u : \Omega \to \mathbb{R}^+$.

Proof:

Consider first the case when $u \in \mathcal{H}(\mathbb{D})$ and u is positive. Then, as we saw in the previous corollary,

$$\frac{\partial u}{\partial z}(z) = \int_0^{2\pi} u(e^{i\theta}) \, \frac{e^{i\theta}}{(e^{i\theta}-z)^2} \, \frac{d\theta}{2\pi}$$

Hence, for $|z| \leqslant \frac{1}{2}$, we obtain

$$\left|\frac{\partial u}{\partial z}(z)\right| \leqslant \int_0^{2\pi} u(e^{i\theta}) \, \frac{1}{|e^{i\theta}-z|^2} \, \frac{d\theta}{2\pi} \leqslant \frac{4}{3}\int_0^{2\pi} u(e^{i\theta}) \, \frac{1-|z|^2}{|e^{i\theta}-z|^2} \, \frac{d\theta}{2\pi} = \frac{4}{3}u(z).$$

Therefore, if $\Delta = \{z : |z-z_o| \leqslant r\} \subset \Omega$ and $\Delta' = \{z : |z-z_o| \leqslant \frac{1}{2}r\}$, we have

$$\left|\frac{\partial u}{\partial z}(z)\right| \leqslant \frac{4}{3r}u(z)$$

for $z \in \Delta'$ and any positive harmonic function on Ω. The compact set K is covered by a finite number of discs like Δ' so the inequality holds (with $c = 4/3\text{dist}(K, \mathbb{C} \setminus \Omega)$). □

Corollary 1.3.6 Harnack's inequality.

For a compact subset K of a domain $\Omega \subset \mathbb{C}$ there is a constant c with

$$u(z_2) \leqslant cu(z_1) \qquad \text{for } z_1, z_2 \in K$$

and for every positive, harmonic function $u : \Omega \to \mathbb{R}^+$.

Proof:

Let Δ be an open disc whose closure lies in Ω. If $z_1, z_2 \in \Delta$ then let γ be the straight line path from z_1 to z_2. Since

$$\begin{aligned}\frac{d}{dt}\log u(\gamma(t)) &= \frac{1}{u(\gamma(t))}\left(\frac{\partial u}{\partial z}(\gamma(t))\gamma'(t) \;+\; \frac{\partial u}{\partial \overline{z}}(\gamma(t))\overline{\gamma'(t)}\right) \\ &= \frac{1}{u(\gamma(t))}2\Re\left(\frac{\partial u}{\partial z}(\gamma(t))\gamma'(t)\right)\end{aligned}$$

we can integrate to obtain

$$\frac{u(z_2)}{u(z_1)} = 2\Re\int_\gamma \frac{1}{u(z)} \, \frac{\partial u}{\partial z}(z) \, dz \leqslant 2\Re \, c \, \text{length}(\gamma)$$

for the constant c of the theorem. Thus $u(z_2) \leqslant c'u(z_1)$ for $c' = 2c \, \text{diameter}(\Delta)$.

Any compact set K can be covered by a finite number of such discs Δ, so the inequality also holds for K. □

Theorem 1.3.7 Harnack's theorem.

If $(u_n : \Omega \to \mathbb{R})$ is an increasing sequence of harmonic functions on a domain $\Omega \subset \mathbb{C}$ then either $u_n(z) \to +\infty$ as $n \to \infty$ at each point of Ω or else the functions u_n converge locally uniformly on Ω to a harmonic function $u : \Omega \to \mathbb{R}$.

Proof:

Let $u(z) = \sup(u_n(z)) \in \mathbb{R} \cup \{+\infty\}$. Then $u_n(z) \to u(z)$ as $n \to \infty$. For a compact subset K of Ω we can apply Harnack's inequality to the positive harmonic functions $u_n - u_m$ for $n > m$ to obtain

$$u_n(z) - u_m(z) \leqslant c\,(u_n(z_o) - u_m(z_o)) \qquad \text{for } z, z_o \in K.$$

Consequently,

$$u(z) - u_m(z) \leqslant c\,(u(z_o) - u_m(z_o)).$$

Therefore, either u is $+\infty$ at each point of Ω or else it is finite at each point. In the latter case we can fix z_o and observe that the above inequalities show that $u_n(z)$ converges uniformly on K by comparison with $u_n(z_o)$. Corollary 1.3.4 shows that the locally uniform limit of the u_n is itself harmonic. □