3.1 Models for the Riemann Sphere.

One dimensional projective complex space $\mathbb{P}(\mathbb{C}^2)$ is the set of all one-dimensional subspaces of $\mathbb{C}^2$. If $\mathbf{z} = (z_1, z_2) \in \mathbb{C}^2 \setminus \mathbf{0}$ then we will denote by $[\mathbf{z}] = [z_1 : z_2]$ the one-dimensional subspace

$$[z_1 : z_2] = \{(\lambda z_1, \lambda z_2) \in \mathbb{C}^2 : \lambda \in \mathbb{C}\}$$

through $\mathbf{z}$. The vector space $\mathbb{C}^2$ has a standard inner product

$$\langle \mathbf{z}, \mathbf{w} \rangle = \overline{z_1} w_1 + \overline{z_2} w_2$$

and associated norm $||\mathbf{z}|| = \sqrt{(|z_1|^2 + |z_2|^2)}$. If $\mathbf{z}, \mathbf{w} \in \mathbb{C}^2 \setminus \mathbf{0}$ then $\mathbf{z}/||\mathbf{z}||$ is a point of unit norm in the subspace $[z_1 : z_2]$ and its distance from the subspace $[w_1 : w_2]$ is

$$d([\mathbf{z}], [\mathbf{w}]) = 2\sqrt{1 - \frac{|\langle \mathbf{z}, \mathbf{w} \rangle|^2}{||\mathbf{z}||^2 ||\mathbf{w}||^2}}.$$

This is a metric on $\mathbb{P}(\mathbb{C}^2)$ called the *Study metric*. With this metric $\mathbb{P}(\mathbb{C}^2)$ becomes a compact Hausdorff space. The two maps

$$\phi : \mathbb{P}(\mathbb{C}^2) \setminus [1:0] \to \mathbb{C} \; ; \; [z_1 : z_2] \mapsto \frac{z_1}{z_2}$$
$$\psi : \mathbb{P}(\mathbb{C}^2) \setminus [0:1] \to \mathbb{C} \; ; \; [z_1 : z_2] \mapsto \frac{z_2}{z_1}$$

are bijections and have $\psi\phi^{-1} : z \mapsto z^{-1}$, so they are charts for a Riemann surface structure on $\mathbb{P}(\mathbb{C}^2)$. We will always assume that $\mathbb{P}(\mathbb{C}^2)$ is made into a Riemann surface in this way.

Exercises

1. Prove that the Study metric is indeed a metric.
2. Show that for $T \in \mathrm{GL}(2, \mathbb{C})$ the map $[\mathbf{z}] \mapsto [T\mathbf{z}]$ is a continuous map from $\mathbb{P}(\mathbb{C}^2)$ to itself. When is it an isometry?
3. If $\mathbf{u}, \mathbf{v}$ is an orthogonal basis for $\mathbb{C}^2$ prove that the map

$$\theta : \mathbb{P}(\mathbb{C}^2) \setminus [\mathbf{u}] \; ; \; [\mathbf{z}] \mapsto \frac{\langle \mathbf{u}, \mathbf{z} \rangle}{\langle \mathbf{v}, \mathbf{z} \rangle}$$

is a chart for the Riemann surface $\mathbb{P}(\mathbb{C}^2)$. What are the transition maps for two such charts?

The map

$$\mathbb{P}(\mathbb{C}^2) \to \mathbb{C}_\infty \; ; \; \begin{cases} [\mathbf{z}] \mapsto \phi(\mathbf{z}) = \dfrac{z_1}{z_2} & \text{if} \quad [\mathbf{z}] \neq [1:0] \\ [0:1] \mapsto \infty \end{cases}$$

is a conformal map which we will use to identify $\mathbb{P}(\mathbb{C}^2)$ with $\mathbb{C}_\infty$. The Study metric induces a metric on $\mathbb{C}_\infty$ called the *chordal metric*:

$$d(z, w) = \frac{2|z - w|}{\sqrt{(1 + |z|^2)}\sqrt{(1 + |w|^2)}} \; , \qquad \text{if } z, w \in \mathbb{C}$$
$$d(z, \infty) = d(\infty, z) = \frac{2}{\sqrt{(1 + |z|^2)}}$$

We can also identify $\mathbb{C}_\infty$ with the unit sphere S^2 in $\mathbb{R}^3$ by using stereographic projection from the point $P = (0,0,1)$. For $z = x + iy \in \mathbb{C}$ the line through P and $(x, y, 0)$ cuts the sphere at P and at the point $\hat{z} = \left(\frac{2x}{1+|z|^2}, \frac{2y}{1+|z|^2}, \frac{1-|z|^2}{1+|z|^2}\right)$. The map

$$\mathbb{C}_\infty \to S^2 \; ; \; \begin{cases} z \mapsto \hat{z} \\ \infty \mapsto P \end{cases}$$

is then a homeomorphism. This makes S^2 into a Riemann surface. Note that the inner product of $\hat{z}, \hat{w} \in S^2$ is

$$\langle \hat{z}, \hat{w} \rangle = \frac{2(\overline{z}w + z\overline{w}) + (1 - |z|^2)(1 - |w|^2)}{(1 + |z|^2)(1 + |w|^2)} = 1 - \frac{2|z - w|^2}{(1 + |z|^2)(1 + |w|^2)}$$

so

$$||\hat{z} - \hat{w}|| = \surd(||\hat{z}||^2 + ||\hat{w}||^2 - 2\langle \hat{z}, \hat{w} \rangle) = \frac{2|z - w|}{\surd(1 + |z|^2)\surd(1 + |w|^2)}.$$

Thus the chordal distance $d(z, w)$ is equal to the length of the chord from $\hat{z}$ to $\hat{w}$ in $\mathbb{R}^3$.

Each of the models $\mathbb{P}(\mathbb{C}^2)$, $\mathbb{C}_\infty$ and S^2 has certain merits. The most elegant theory uses $\mathbb{P}(\mathbb{C}^2)$; while S^2 is easy to visualize and $\mathbb{C}_\infty$ is often easy for calculations. We will switch from one to another freely.

Exercises

4. [This assumes a little knowledge of algebraic geometry.] Let $\mathbf{z} \in \mathbb{C}^N$ be a row vector. Then $\mathbf{z}^*\mathbf{z} = \overline{\mathbf{z}}^t\mathbf{z}$ is in the <u>real</u> vector space $\mathrm{Her}(N)$ of Hermitian matrices. What is the dimension of the real projective space $\mathbb{P}(\mathrm{Her}(N))$? Show that

$$J : \mathbb{P}(\mathbb{C}^N) \to \mathbb{P}(\mathrm{Her}(N)) \; ; \; [\mathbf{z}] \mapsto [\mathbf{z}^*\mathbf{z}]$$

is a well defined, injective map and that its image is a projective variety (i.e. the set where a collection of homogeneous polynomials vanish). When $N = 2$, the image is a conic in $\mathbb{P}(\mathbb{R}^4)$ isomorphic to the sphere. [Thus J generalizes the identification of $\mathbb{P}(\mathbb{C}^2)$ with S^2.]

If $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbb{C})$ then T induces a map

$$\mathbb{P}(T) : \mathbb{P}(\mathbb{C}^2) \to \mathbb{P}(\mathbb{C}^2) \; ; \; [z_1 : z_2] \mapsto [az_1 + bz_2 : cz_1 + dz_2].$$

It corresponds to the Möbius transformation $\mathbb{C}_\infty \to \mathbb{C}_\infty$; $z \mapsto (az + b)/(cz + d)$. Therefore the map

$$\mathrm{GL}(2, \mathbb{C}) \to \text{Möb} \; ; \; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left\{ z \mapsto \frac{az + b}{cz + d} \right\}$$

is a group homomorphism onto the group Möb of Möbius transformations. Its kernel is $\{\lambda I : \lambda \in \mathbb{C}^\times\}$ so Möb is isomorphic to the quotient $\mathrm{GL}(2, \mathbb{C})/\mathbb{C}^\times I$, which is called the *projective general linear group* $\mathrm{PGL}(2, \mathbb{C})$. Similarly, Möb is isomorphic to the *projective special linear group* $\mathrm{PSL}(2, \mathbb{C}) = \mathrm{SL}(2, \mathbb{C})/\{-I, +I\}$.

3.2 Rational Functions.

Let $f : R \to \mathbb{C}_\infty$ be a meromorphic function on a Riemann surface R. A point $z_o \in R$ is a *pole* of f if $f(z_o) = \infty$. By Proposition 2.2.1 these are isolated. If R is a domain in $\mathbb{C}$ then f will have a Laurent series $\sum_{n=-N}^{\infty} a_n(z - z_o)^n$ which converges on a neighbourhood of z_o. The coefficient N is equal to $\deg f(z_o)$ and is called the *order* of the pole. The sum $\sum_{n=-N}^{-1} a_n(z - z_o)^n$ is called the *principal part* of f at z_o. It is a polynomial in $(z - z_o)^{-1}$ and the difference between f and its principal part is an analytic map into $\mathbb{C}$ on a neighbourhood of z_o. Similarly, if R is a domain in $\mathbb{C}_\infty$ and ∞ is a pole of f then f has a Laurent series $\sum_{n=-N}^{\infty} a_n z^{-n}$ convergent in a neighbourhood of ∞. The sum $\sum_{n=-N}^{-1} a_n z^{-n}$ is the principal part of f at ∞. It is a polynomial in z.

A *rational function* r is the quotient a/b of two polynomials a and b which have no common zeros. It is therefore a meromorphic function from $\mathbb{C}_\infty$ to itself. If the polynomials a and b have degrees $\deg a$ and $\deg b$ respectively, then r will have $\deg a$ zeros in $\mathbb{C}$ (counting multiplicity) and a zero of order $\deg b - \deg a$ at ∞ if $\deg b > \deg a$. Therefore r has degree $\max(\deg a, \deg b)$.

Theorem 3.2.1

A function $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is meromorphic if, and only if, it is rational.

Proof:

It is clear that a rational function is meromorphic. Suppose that f is meromorphic. Then its poles are isolated in the compact set $\mathbb{C}_\infty$, so there are only finitely many of them, say $z_1, z_2, \ldots, z_K$. Let p_k be the principal part of f at the pole z_k. Then $g = f - \sum p_k$ is a meromorphic function and it has no poles. By theorem 2.2.3 g must be constant. Hence f is rational. □

Exercises

5. A *divisor* on a <u>compact</u> Riemann surface is a function $d : R \to \mathbb{Z}$ which is zero except at a finite set of points. These form a commutative group $\mathcal{D}$. The map

$$\delta : \mathcal{D} \to \mathbb{Z} \quad ; \quad d \mapsto \sum (d(z) : z \in R)$$

is a homomorphism. Let $\mathcal{D}_0$ be its kernel.

(a) Let f be a meromorphic function on R which is not identically zero, so $f \in \mathcal{M}(R)^\times$. Then f has finitely many zeros and poles. Let (f) be the divisor which is $\deg f(z)$ at any zero z, $-\deg f(z)$ at any pole z, and zero elsewhere. Show that this gives a homomorphism of commutative groups

$$\mathcal{M}(R)^\times \to \mathcal{D}_0 \quad ; \quad f \mapsto (f).$$

Find the kernel of this homomorphism. The quotient $\mathcal{D}/\{(f) : f \in \mathcal{M}(R)^\times\}$ is called the *divisor class group* of R.

(b) Show that the divisor class group of $\mathbb{C}_\infty$ is trivial.

6. Find all the meromorphic 1-forms (differentials) on $\mathbb{C}_\infty$.

3.3 Möbius Transformations

Theorem 3.3.1

$\text{Aut}\,\mathbb{C}_\infty = $ *Möb* .

Proof:

If $f \in \text{Aut}\,\mathbb{C}_\infty$ then $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is meromorphic and bijective. Hence Theorem 3.2.1 shows that it is a rational function of degree 1. These are precisely the Möbius transformations $z \mapsto (az+b)/(cz+d)$ for $ad - bc \neq 0$. □

If z_0, z_1, z_∞ are three distinct points of $\mathbb{C}_\infty$ then there is a unique Möbius transformation T which maps them to $0, 1, \infty$ respectively. It is given by

$$z \mapsto \frac{z - z_0}{z - z_\infty} \frac{z_1 - z_\infty}{z_1 - z_0}.$$

The image of $z \in \mathbb{C}_\infty$ under this transformation is called the *cross ratio* $\mathcal{R}(z_0, z_1, z_\infty, z)$. It is then clear that the following result is true.

Proposition 3.3.2 Cross ratios

There is a Möbius transformation which maps the four distinct points z_0, z_1, z_∞, z in $\mathbb{C}_\infty$ onto the distinct points w_0, w_1, w_∞, w, in order, if and only if $\mathcal{R}(z_0, z_1, z_\infty, z) = \mathcal{R}(w_0, w_1, w_\infty, w)$.

□

Let $T : z \mapsto (az+b)/(cz+d)$ be a Möbius transformation with $ad - bc = \delta \neq 0$. Then T fixes a point $z \in \mathbb{C}$ if, and only if, $az^2 + (d-a)z - b = 0$, and fixes ∞ if, and only if, $c = 0$. Thus T is either the identity or it fixes exactly 1 or 2 points of $\mathbb{C}_\infty$.

Theorem 3.3.3

If $\pi : \mathbb{C}_\infty \to R$ is a universal covering of the Riemann surface R, then π is conformal.

Proof:

Theorem 2.3.5 showed that R was the quotient of $\mathbb{C}_\infty$ by a subgroup G of $\text{Aut}\,\mathbb{C}_\infty$. Moreover every element of G other than the identity has no fixed points. We have seen that there are no such automorphisms. □

Suppose that T has exactly two fixed points z_0 and z_∞. Then we can find a Möbius transformation S which maps z_0 and z_∞ to 0 and ∞ respectively. So $T_1 = STS^{-1}$ is a Möbius transformation which fixes 0 and ∞ alone. Hence we must have $T_1 = STS^{-1} : z \mapsto \lambda z$ for some $\lambda \in \mathbb{C} \setminus \{0, 1\}$. Now, if T_1 and T_2 are conjugate in Möb, say $T_2 = UT_1U^{-1}$, then U must map the fixed points of T_2 to the fixed points of T_1. Hence, $z \mapsto \lambda z$ and $z \mapsto \mu z$ are conjugate if, and only if, $\mu = \lambda$ or λ^{-1}. It is easy to find the value of λ from T. For, if $T : z \mapsto (az+b)/(cz+d)$ then the matrix $M(T) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is determined by T up to multiplication of each entry by a non-zero complex number. Hence $\tau(T) = (\text{tr}\, M(T))^2 / \det M(T)$ does depend only on T. Since the trace and determinant are invariant under conjugation we see that

$$\tau(T) = \tau(T_1) = \frac{(\lambda+1)^2}{4\lambda} = \tfrac{1}{4}(\lambda + \lambda^{-1}) + \tfrac{1}{2}.$$

Thus $\tau(T)$ determines the pair (λ, λ^{-1}) and this determines the conjugacy class of T in the group of Möbius transformations. We give names to various different classes of transformations:

T is a Möbius transformation not equal to the identity.

T is *elliptic*	$\Leftrightarrow$	$|\lambda| = 1$ but $\lambda \neq 1$	$\Leftrightarrow$	$\tau(T) \in [0, 1)$
T is *hyperbolic*	$\Leftrightarrow$	$\lambda \in \mathbb{R} \setminus \{-1, 0, 1\}$	$\Leftrightarrow$	$\tau(T) \in (1, \infty)$
T is *loxodromic*	$\Leftrightarrow$	$\lambda \in \mathbb{C} \setminus \mathbb{R}$ and $|\lambda| \neq 1$	$\Leftrightarrow$	$\tau(T) \in \mathbb{C} \setminus [0, \infty)$

If T has exactly one fixed point z_∞ then we can conjugate T by a Möbius transformation S which sends z_∞ to ∞ . Then $T_1 = STS^{-1}$ fixes only ∞ and so is $z \mapsto z + \nu$ for ν a non-zero complex number. All such Möbius transformations T_1 are conjugate to one another. In this case we say that T is *parabolic*. Note that $\tau(T) = 1$ if, and only if, T is either parabolic or the identity.

Exercises

Let $T : z \mapsto (az + b)/(cz + d)$ be a Möbius transformation.

7. Consider the chordal metric on $\mathbb{C}_\infty$ and show that T multiplies the length of an infinitesimally short curve at z by the factor

$$\frac{|T'(z)|(1 + |z|^2)}{1 + |T(z)|^2} = \frac{|ad - bc|(1 + |z|^2)}{|az + b|^2 + |cz + d|^2}.$$

Show that the maximum and minimum values of this quantity are

$$s + \sqrt{s^2 - 1} \qquad \text{and} \qquad s - \sqrt{s^2 - 1}$$

where

$$s = \frac{|a|^2 + |b|^2 + |c|^2 + |d|^2}{2|ad - bc|}.$$

[Hint: Think about $\mathbb{C}_\infty$ as $\mathbf{P}(\mathbb{C}^2)$.]

8. Let $Z(T) = \{S \in \text{Möb} : ST = TS\}$.
   (a) Show that $Z(T)$ is a subgroup of Möb.
   (b) Find which groups (up to isomorphism)can arise as $Z(T)$ for some Möbius transformation T .

9. Let A be a 2×2 complex matrix with trace equal to 0. Show that the series

$$\exp A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n$$

converges and prove the following properties.
   (a) If $AB = BA$ then $\exp(A + B) = \exp A \exp B$.
   (b) $\{\exp tA : t \in \mathbb{R}\}$ is a commutative group under multiplication of matrices.
   (c) The function $f(t) = \det \exp tA$ satisfies $f'(t) = f(t) \operatorname{tr} A = 0$. Hence $\exp tA \in SL(2, \mathbb{C})$.

Let $\exp tA$ now denote the Möbius transformation determined by the matrix $\exp tA$. Show that every Möbius transformation is equal to $\exp A$ for some matrix A . Is the choice of A unique? For $z \in \mathbb{C}_\infty$ the images of z under the Möbius transformations $\exp tA$ for $t \in \mathbb{R}$ trace out a curve. Which curves can arise in this way? Sketch examples. (The groups $\{\exp tA : t \in \mathbb{R}\}$ for some A are the 1-parameter subgroups of the Lie group Möb.)
