4 4 THE COMPLEX PLANE

4.1 Meromorphic functions.

A *entire* function is an analytic function from the complex plane to itself. Suppose that $f : \mathbb{C} \to \mathbb{C}_\infty$ is a meromorphic function. Then it will have a finite or infinite sequence of poles (z_n). These are isolated so, if there are infinitely many, they must converge to ∞. The following theorem shows that any such sequence of poles can occur.

Theorem 4.1.1 Mittag-Leffler expansions

Let (z_n) be a sequence of points in $\mathbb{C}$ which is either finite or else converges to ∞. For each n let p_n be a polynomial. Then there is a meromorphic function $f : \mathbb{C} \to \mathbb{C}_\infty$ which has a pole at each z_n with principal part $p_n((z - z_n)^{-1})$ and no other poles. Any two such functions differ by an entire function.

Proof:

For any polynomial q_n the function $p_n((z - z_n)^{-1}) - q_n(z)$ has the same principal part at z_n as $p_n((z - z_n)^{-1})$. We will show that we can choose the q_n so that the series $\sum p_n((z - z_n)^{-1}) - q_n(z)$ converges locally uniformly. The function it converges to will then have the required properties. If two functions f_1 and f_2 have these properties then their difference has no poles and so is entire.

If there are only finitely many poles then we can take each q_n equal to 0. The finite sum $\sum p_n((z - z_n)^{-1}) - q_n(z)$ clearly gives a rational function with the desired behaviour at each pole. From now on we will assume that the sequence (z_n) is infinite and converges to ∞. Let (M_n) be a sequence of positive numbers with $\sum M_n$ finite. For each n the function $p_n((z - z_n)^{-1})$ is analytic on the disc $\{z : |z| < |z_n|\}$ so its Taylor series converges uniformly on the disc $\{z : |z| \leqslant \frac{1}{2}|z_n|\}$. Take q_n to be a partial sum of this Taylor series with

$$|p_n((z - z_n)^{-1}) - q_n(z)| \leqslant M_n \qquad \text{for} \qquad |z| \leqslant \tfrac{1}{2}|z_n|.$$

For each $R > 0$ there are only finitely many n with $|z_n| < R$. The finite sum $\sum \big(p_n((z - z_n)^{-1}) - q_n(z) : |z_n| < R\big)$ therefore gives a rational function which has the correct principal parts at each z_n with $|z_n| < R$ and no other poles. The sum $\sum \big(p_n((z - z_n)^{-1}) - q_n(z) : |z_n| \geqslant R\big)$ converges uniformly on $\{z : |z| \leqslant \frac{1}{2}R\}$ by comparison with $\sum M_n$. So it gives an analytic function on $\{z : |z| \leqslant \frac{1}{2}R\}$. Since R is arbitrary, the full series $\sum p_n((z-z_n)^{-1}) - q_n(z)$ converges giving a meromorphic function with poles at each z_n having principal part $p_n((z-z_n)^{-1})$ and no other poles. □

Exercises

-1. Give an example to show that the series $\sum p_n((z - z_n)^{-1})$ in the theorem need not converge.

2. Show that any sequence of points (z_n) in $\mathbb{D}$ with $|z_n| \to 1-$ as $n \to \infty$ is the sequence of poles of a meromorphic function $f : \mathbb{D} \to \mathbb{C}_\infty$.

4.2 Entire functions.

Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. If f has no zeros then the monodromy theorem 2.3.2 shows that we may find an entire function g with $f = \exp g$. If f has finitely many zeros $z_1, z_2, \dots, z_N$, each repeated according to its multiplicity, then

$$f(z) = F(z) \prod_{n=1}^{N} (z - z_n)$$

for an entire function F with no zeros. We wish to find a similar formula when f has infinitely many zeros. To do this we will need to consider functions defined by infinite products.

Let (u_n) be a sequence of non-zero complex numbers. We will say that the infinite product $\prod_{n=1}^{\infty} u_n$ converges to $L \neq 0$ if the sequence of partial products $L_N = \prod_{n=1}^{N} u_n$ con-verges to L as $N \to \infty$. For this to happen we must have $u_n \to 1$ so it is convenient to write $u_n = 1 + a_n$. If $a_n \to 0$ then there will be a N_o with $|a_n| < 1$ for $n > N_o$. Let $\mathrm{Log} : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ be the principal branch of the logarithm. Then

$$L_N = L_{N_o} \prod_{n=N_o+1}^{N} (1 + a_n) = \exp \sum_{n=N_o+1}^{N} \mathrm{Log}(1 + a_n)$$

for $N > N_o$. Hence the product $\prod_{n=1}^{\infty}(1 + a_n)$ converges if, and only if, $a_n \to 0$ and the series $\sum_{n=N_o}^{\infty} \mathrm{Log}(1 + a_n)$ converges. This enables us to transfer results about series to products. For any sequence of complex numbers u_n, including 0 , we say that the product $\prod u_n$ converges if there exists n_o with $u_n \neq 0$ for $n \geqslant n_o$ and $\prod_{n=n_o}^{\infty} u_n$ converges.

Note in particular that $\mathrm{Log}(1 + a)$ is asymptotic to a as $a \to 0$ so the series $\sum_{n=N_o}^{\infty} \mathrm{Log}(1 + a_n)$ converges absolutely if, and only if, the series $\sum |a_n|$ converges. Suppose that $(a_n : \Omega \to \mathbb{C})$ is a sequence of analytic functions on the domain Ω and that $\sum M_n$ is a convergent series. If $|a_n(z)| < M_n$ for $z \in \Omega$, then the series $\sum |a_n(z)|$ converges uniformly and $a_n(z)$ converges uniformly to 0. Consequently the series $\sum_{n=n_o}^{\infty} \mathrm{Log}(1 + a_n(z))$ will converge uniformly to an analytic function for n_o large enough. This proves that the product $\prod(1 + a_n(z))$ converges on Ω to an analytic function which has zeros at the points where $(1 + a_n(z)) = 0$ for some n.

If (z_n) is an infinite sequence of points in $\mathbb{C}$ which converges to ∞ then the product

$$\prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right)$$

need not converge. However, if $\sum |z_n|^{-1}$ converges, then the product will converge to an entire function with zeros precisely at the points z_n. To deal with sequences (z_n) which have $\sum |z_n|^{-1}$ divergent we need to introduce exponential factors into the product.

Theorem 4.2.1 Weierstrass products

Let (z_n) be a sequence of points in $\mathbb{C}$ which is either finite or else tends to ∞. Then there is an entire function f which has a zero at each point ζ in the sequence with order equal to the number of times that it occurs in the sequence, and no other zeros. If g is another such function then $f(z) = g(z) \exp h(z)$ for some entire function h.

Proof:

Choose positive numbers M_n for which $\sum M_n$ converges. The function $z \mapsto \mathrm{Log}\left(1 - \frac{z}{z_n}\right)$ is analytic on $\{z : |z| < |z_n|\}$ so its Taylor series

$$-\frac{z}{z_n} - \frac{1}{2}\left(\frac{z}{z_n}\right)^2 - \frac{1}{3}\left(\frac{z}{z_n}\right)^3 - \dots$$

converges uniformly on $\{z : |z| \leqslant \frac{1}{2}|z_n|\}$. Hence we can choose natural numbers $N(n)$ so that

$$q_n(z) = \frac{z}{z_n} + \frac{1}{2}\left(\frac{z}{z_n}\right)^2 + \frac{1}{3}\left(\frac{z}{z_n}\right)^3 + \ldots + \frac{1}{N(n)}\left(\frac{z}{z_n}\right)^{N(n)}$$

satisfies

$$\left|\mathrm{Log}\left(1 - \frac{z}{z_n}\right) + q_n(z)\right| \leqslant M_n \qquad \text{for} \qquad |z| \leqslant \tfrac{1}{2}|z_n|.$$

Therefore, the series

$$\sum_{n=1}^{\infty} \left(\mathrm{Log}\left(1 - \frac{z}{z_n}\right) + q_n(z)\right)$$

will converge locally uniformly. Hence,

$$f(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) \exp q_n(z)$$

converges and gives an entire function f with the desired properties.

If g were another such function then g/f would be an entire function with no zeros and therefore equal to $\exp h$ for some entire function h. □

Corollary 4.2.2

Every meromorphic function $f : \mathbb{C} \to \mathbb{C}_\infty$ is the quotient a/b of two entire functions a and b.

Proof:

The theorem enables us to construct an entire function b whose zeros are poles of f. Then $a = b.f$ is also entire. □

As an example, let us try to construct a entire function with zeros at the integer points. The series $\sum n^{-2}$ converges so the proof of Weierstrass theorem shows that

$$f(z) = z \prod_{n \neq 0} \left(1 - \frac{z}{n}\right) e^{z/n}$$

converges to the desired entire function. We can rewrite this series as

$$f(z) = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

Because of the locally uniform convergence we can differentiate the product to obtain

$$\begin{aligned} f'(z) &= f(z) \left\{ \frac{1}{z} + \sum_{n \neq 0} \left(\frac{1}{z-n} + \frac{1}{n} \right) \right\} \\ &= f(z) \left\{ \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{2z}{z^2 - n^2} \right) \right\} \end{aligned}$$

Hence $f'(z) = f(z)\varepsilon_1(z) = f(z)\pi \cot \pi z$. We also have $f'(0) = 1$ so we can solve this differential equation to obtain

$$z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) = f(z) = \frac{\sin \pi z}{\pi}.$$

Exercises

3. Show that the product

$$g(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{n}\right) e^{z/n}$$

converges and satisfies

$$g'(z) = g(z) \sum_{n=1}^{\infty} \left(\frac{1}{z-n} + \frac{1}{n} \right).$$

Deduce that $g(z+1) = -zg(z)e^{\gamma}$ for some constant γ and prove that

$$\gamma = \lim_{N \to \infty} \sum_{n=1}^{N} \frac{1}{n} - \log N.$$

(This is Euler's constant.)

4.3 Quotients of the complex plane.

Theorem 4.3.1

The group $\mathrm{Aut}\,\mathbb{C}$ *consists of the maps* $z \mapsto az + b$ *for* $a \in \mathbb{C} \setminus \{0\}$ *and* $b \in \mathbb{C}$.

Proof:

Suppose that $T : \mathbb{C} \to \mathbb{C}$ is conformal. Then we can consider it acting on $\mathbb{C}_\infty$ with an isolated singularity at ∞ and show that it has a removable singularity there. The set $U = T^{-1}(\mathbb{D})$ is open in $\mathbb{C}$ and T maps every point of $\mathbb{C} \setminus U$ into $\{z \in \mathbb{C} : |z| \geqslant 1\}$. Hence the map $S : z \mapsto 1/T(z^{-1})$ is bounded on a neighbourhood of 0 and so must have a removable singularity there. Consequently T extends to an analytic map $T : \mathbb{C}_\infty \to \mathbb{C}_\infty$. We know from Theorem 3.2.1 that T must be a rational function and the only ones which restrict to give a conformal map $\mathbb{C} \to \mathbb{C}$ are those of the form $z \mapsto az + b$ with $a \in \mathbb{C} \setminus \{0\},\ b \in \mathbb{C}$. □

Suppose that G is a subgroup of $\mathrm{Aut}\,\mathbb{C}$ for which the quotient $\mathbb{C}/G$ is a Riemann surface. Then Theorem 2.3.6 shows that every element of $G \setminus \{I\}$ has no fixed points. The only maps $z \mapsto az + b$ which have this property are those with $a = 1$; the translations. Thus G is a subgroup of the group of translations: $\{z \mapsto z + b : b \in \mathbb{C}\}$. The set $\Lambda = \{T(0) : T \in G\}$ is then an additive subgroup of $\mathbb{C}$ isomorphic to G. For $\mathbb{C}/G$ to be a Riemann surface we certainly need $0 \in \mathbb{C}$ to be isolated in $\Lambda = G(0)$ so there is a $\delta > 0$ with $|\lambda| > 2\delta$ for each $\lambda \in \Lambda \setminus \{0\}$. Conversely, if this is true, then the neighbourhood $U = \{z \in \mathbb{C} : |z - w| < \delta\}$ of any point $w \in \mathbb{C}$ has all the sets $T(U)$ for $T \in G$ disjoint, so $\mathbb{C}/G$ is a Riemann surface by Theorem 2.3.6.

We will often identify G with Λ and write $\mathbb{C}/\Lambda$ for $\mathbb{C}/G$. We have shown that this quotient is a Riemann surface if $\inf(|\lambda| : \lambda \in \Lambda \setminus \{0\}) > 0$. Any additive subgroup of $\mathbb{C}$ with this property is called a *lattice* in $\mathbb{C}$.

Theorem 4.3.2

A subset Λ *of* $\mathbb{C}$ *is a lattice if, and only if, it is of one of the three forms:*

(a) $\{0\}$.

(b) $\mathbb{Z}\omega_1 = \{n\omega_1 : n \in \mathbb{Z}\}$ *for some* $\omega_1 \in \mathbb{C} \setminus \{0\}$.

(c) $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \{n\omega_1 + m\omega_2 : n, m \in \mathbb{Z}\}$ *for some* $\omega_1, \omega_2 \in \mathbb{C}$ *which are linearly independent over* $\mathbb{R}$.

In these three cases we have:

(a) $\mathbb{C}/\{0\} = \mathbb{C}$.

(b) $\mathbb{C}/\mathbb{Z}\omega_1$ *is conformally equivalent to the infinite cylinder* $\mathbb{C} \setminus \{0\}$.

(c) $\mathbb{C}/(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2)$ *is a compact Riemann surface homeomorphic to a torus.*

In case (c) we call $\mathbb{C}/(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2)$ an *analytic torus*. There are many conformally different analytic tori.

Proof:

If $\Lambda = \{0\}$ then (a) holds and $\mathbb{C}/\{0\}$ is clearly $\mathbb{C}$. Otherwise we can choose $\omega \in \Lambda \setminus \{0\}$ with $|\omega|$ smallest. Let this be ω_1. If $\Lambda = \mathbb{Z}\omega_1$ then (b) holds and the mapping

$$\mathbb{C}/\mathbb{Z}\omega_1 \to \mathbb{C} \setminus \{0\} \quad ; \quad [z] \mapsto \exp(2\pi i z/\omega_1)$$

is conformal. Otherwise we can choose $\omega \in \Lambda \setminus \mathbb{Z}\omega_1$ with $|\omega|$ smallest. Let this be ω_2.

Suppose that ω_1, ω_2 were not linearly independent over $\mathbb{R}$. Then $\omega_2 = x\omega_1$ for some $x \in \mathbb{R}$. We can write $x = n + q$ with $n \in \mathbb{Z}$ and $0 \leqslant q < 1$. Then $\omega_2 - n\omega_1 = q\omega_1 \in \Lambda$. The definition of ω_1 implies that q must be zero and then this contradicts $\omega_2 \notin \mathbb{Z}\omega_1$. Hence ω_1, ω_2 are linearly independent. If $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ then (c) holds. The space $\mathbb{C}/(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2)$ is easily seen to be homeomorphic to the space obtained by identifying the opposite sides of the *fundamental parallelogram* $P = \{x\omega_1 + y\omega_2 : 0 \leqslant x, y \leqslant 1\}$. This is clearly a torus.

It remains to show that we cannot have any elements $\omega \in \Lambda \setminus (\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2)$. Suppose that we did, then $\omega = x\omega_1 + y\omega_2$ for some $x, y \in \mathbb{R}$. We can choose $n, m \in \mathbb{Z}$ with $|x - n|, |y - m| \leqslant \frac{1}{2}$. Then

$$|\omega - (n\omega_1 + m\omega_2)| = |(x - n)\omega_1 + (y - m)\omega_2|.$$

The triangle inequality shows that this is less than

$$\tfrac{1}{2}|\omega_1| + \tfrac{1}{2}|\omega_2| \leqslant |\omega_1|.$$

and the inequality must be strict because ω_1, ω_2 are linearly independent over $\mathbb{R}$. This contradicts the definition of ω_1. □

Exercises

4. A function $f : \mathbb{C} \to \mathbb{C}$ is *periodic with period* p if $f(z + p) = f(z)$ for every $z \in \mathbb{C}$. Show that the set of periods of an analytic function f is either a lattice in $\mathbb{C}$ or else all of $\mathbb{C}$.
5. Show that every analytic function $f : \mathbb{C} \to \mathbb{C}$ which is periodic with a period $p \neq 0$ has a Fourier expansion $f(z) = \sum_{n=-\infty}^{\infty} a_n \exp(2\pi i n z/p)$ convergent everywhere.
6. Show that for any subset E of $\mathbb{C} \setminus \{0\}$ which has no accumulation points except possibly 0 or ∞ there is a meromorphic function on $\mathbb{C} \setminus \{0\}$ with poles precisely at the points of E.
