6 6 DISCRETE GROUPS

6.1 Discontinuous Group Actions

Let G be a subgroup of Möb($\mathbb{D}$). This group *acts discontinuously on* $\mathbb{D}$ if, for every compact subset K of $\mathbb{D}$, the set $\{T \in G : T(K) \cap K \neq \emptyset\}$ is finite.

Proposition 6.1.1

Let G be a subgroup of Möb($\mathbb{D}$) and $z_o \in \mathbb{D}$. Then G acts discontinuously on $\mathbb{D}$ if and only if

$$\{T \in G : \rho(z_o, T(z_o)) \leqslant r\}$$

is finite for every $r < \infty$.

Proof:

The closed disc $C = \{z \in \mathbb{D} : \rho(z_o, z) \leqslant r\}$ is compact and

$$\begin{aligned} T(C) \cap C \neq \emptyset \quad &\Leftrightarrow \quad \text{there exists } w \in \mathbb{D} \text{ with } \rho(z_o, w) \leqslant r \text{ and } \rho(T(z_o), w) \leqslant r \\ &\Leftrightarrow \quad \rho(z_o, T(z_o)) \leqslant 2r \end{aligned}$$

Suppose first that G acts discontinuously on $\mathbb{D}$. Then the set $\{T \in G : C \cap T(C) \neq \emptyset\}$ is finite and hence $\{T \in G : \rho(z_o, T(z_o)) \leqslant 2r\}$ is also finite.

Now suppose that $\{T \in G : \rho(z_o, T(z_o)) \leqslant r\}$ is finite for every $r < \infty$. Any compact subset K of $\mathbb{D}$ must lie within some closed ball $C = \{z \in \mathbb{D} : \rho(z_o, z) \leqslant r\}$. Then

$$\{T \in G : K \cap T(K) \neq \emptyset\} \subset \{T \in G : C \cap T(C) \neq \emptyset\} = \{T \in G : \rho(z_o, T(z_o)) \leqslant 2r\} .$$

So G acts discontinuously on $\mathbb{D}$. $\square$

Proposition 6.1.2

Let G act discontinuously on $\mathbb{D}$ and let $z_o \in \mathbb{D}$. Then

The stabilizer Stab(z_o) $= \{T \in G : T(z_o) = z_o\}$ is a finite, cyclic subgroup of G.

The orbit $G(z_o)$ is a closed, discrete subset of $\mathbb{D}$. So there is a $\delta > 0$ with

$$\rho(z_o, T(z_o)) > 4\delta \qquad \text{for all } T \in G \setminus \mathrm{Stab}(z_o) .$$

Proof:

The stabilizer $\mathrm{Stab}(z_o) = \{T \in G : \{z_o\} \cap T(\{z_o\}) \neq \emptyset\}$ is certainly finite and a subgroup of $\{T \in \text{Möb}(\mathbb{D}) : T(z_o) = z_o\}$. Hence it is conjugate to a finite subgroup of

$$\{T \in \text{Möb}(\mathbb{D}) : T(0) = 0\} = \{z \mapsto \omega z : |\omega| = 1\} .$$

Such a subgroup must be cyclic since it is generated by the rotation through the smallest positive angle. Hence $\mathrm{Stab}(z_o)$ is generated by an elliptic transformation of finite order that fixes z_o.

For $w \in \mathbb{D}$ we have

$$\rho(z_o, T(z_o)) \leqslant \rho(w, z_o) + \rho(w, T(z_o)) \ .$$

So Proposition 6.1.1 shows that $\rho(w, T(z_o)) \leqslant r$ for only a finite number of $T \in G$. Consequently, the orbit $G(w)$ is closed and discrete in $\mathbb{D}$. $\square$

For the remainder of this section G will be a subgroup of Möb($\mathbb{D}$) that acts discontinuously on $\mathbb{D}$. The quotient $\mathbb{D}/G$ is the set of all G-orbits in $\mathbb{D}$ and $q : \mathbb{D} \to \mathbb{D}/G$ is the quotient map, which sends a point z to the orbit $G(z)$. We wish to show that the quotient can be made into a Riemann surface in such a way that the quotient map is holomorphic.

Example:

First consider the example where G is the cyclic group

$$\{z \mapsto e^{2\pi ik/N}z : k = 0, 1, 2, \ldots, N-1\}$$

of order N that fixes 0. The map

$$\pi : \mathbb{D} \to \mathbb{D} \ ; \quad z \mapsto z^N$$

is constant on each orbit and maps each orbit to a distinct point. Hence it induces a bijection

$$\phi : \mathbb{D}/G \to \mathbb{D} \ ; \quad q(z) \mapsto z^N \ .$$

We could then make $\mathbb{D}/G$ into a Riemann surface by insisting that ϕ is conformal. The quotient map will then be holomorphic since $z \mapsto z^N$ is.

A very similar example is when G is the cyclic group of order N that fixes another point $z_o \in \mathbb{D}$. We may conjugate by the Möbius transformation

$$z \mapsto \frac{z - z_o}{1 - \overline{z_o}z}$$

to reduce this to the previous example. Hence we see that the quotient is identified with $\mathbb{D}$ by taking the quotient mapping to be

$$q : \mathbb{D} \to \mathbb{D} \ ; \quad z \mapsto \left(\frac{z - z_o}{1 - \overline{z_o}z}\right)^N \ .$$

The general case arises by using these examples to construct charts on $\mathbb{D}/G$ that make it into a Riemann surface.

First we will define a metric $\widetilde{\rho}$ on the quotient. For orbits $p_j = q(z_j)$ set

$$\begin{aligned}\widetilde{\rho}(p_1, p_2) &= \inf\{\rho(w_1, w_2) : w_1 \in p_1, w_2 \in p_2\} \\ &= \inf\{\rho(T_1(z_1), T_2(z_2)) : T_1, T_2 \in G\} \ .\end{aligned}$$

It is simple to see that $\widetilde{\rho}$ is non-negative, symmetric and satisfies the triangle inequality. Since each $T \in G \subset$ Möb($\mathbb{D}$) is an isometry for the hyperbolic metric ρ, we have

$$\widetilde{\rho}(p_1, p_2) = \inf\{\rho(z_1, T(z_2)) : T \in G\} \ .$$

Now Proposition 6.1.2 shows that the orbit $G(z_2)$ is closed, so we see that $\widetilde{\rho}(p_1, p_2) = 0$ only when $p_1 = p_2$. Hence, $\widetilde{\rho}$ is a metric on the quotient $\mathbb{D}/G$. It gives the quotient topology on $\mathbb{D}/G$, which is therefore Hausdorff.

Lemma 6.1.3

Let G be a subgroup of Möb($\mathbb{D}$) that acts discontinuously on $\mathbb{D}$ and let $z_o \in \mathbb{D}$ be fixed by only the identity transformation in G. Then there is a $\delta = \delta_{z_o} > 0$ so the quotient map restricts to give an isometry

$$q| : \mathbb{D}(z_o, \delta) \to \widetilde{\mathbb{D}}(q(z_o), \delta)$$

from the disc $\mathbb{D}(z_o, \delta)$ of radius δ about z_o in $\mathbb{D}$ onto the disc $\widetilde{\mathbb{D}}(q(z_o), \delta)$ of radius δ about $q(z_o)$ in $\mathbb{D}/G$.

Proof:

Choose δ as in Proposition 6.1.2 so that $\rho(z_o, T(z_o)) > 4\delta$ for each $T \in G \setminus \{I\}$.

If $p \in \widetilde{\mathbb{D}}(q(z_o), \delta)$, then $\widetilde{\rho}(q(z_o), p) < \delta$. Hence, there is a $z \in \mathbb{D}(z_o, \delta)$ with $q(z) = p$. If z' is any other point of the orbit p, then $z' = T(z)$ for some $T \in G \setminus \{I\}$. Hence,

$$\rho(z_o, z') \geqslant \rho(z_o, T(z_o)) - \rho(T(z_o), T(z)) = \rho(z_o, T(z_o)) - \rho(z_o, z) > 4\delta - \delta$$

and so $z' \notin \mathbb{D}(z_o, \delta)$. Therefore, the restriction

$$q| : \mathbb{D}(z_o, \delta) \to \widetilde{\mathbb{D}}(q(z_o), \delta)$$

is bijective.

Now suppose that $z_1, z_2 \in \mathbb{D}(z_o, \delta)$. For each $T \in G \setminus \{I\}$ we have

$$\begin{aligned} \rho(z_1, T(z_2)) &\geqslant \rho(z_o, T(z_o)) - \rho(z_o, z_1) - \rho(T(z_o), T(z_2)) \\ &= \rho(z_0, T(z_o)) - \rho(z_o, z_1) - \rho(z_o, z_2) \\ &> 4\delta - \delta - \delta = 2\delta\ . \end{aligned}$$

Hence,

$$\widetilde{\rho}(q(z_1), q(z_2)) = \inf\{\rho(z_1, T(z_2)) : T \in G\} = \rho(z_1, z_2)$$

and the restriction of q is an isometry. □

This lemma deals with the case where the stabilizer of a point z_o is trivial. Otherwise, the stabilizer of z_o is a cyclic group of some order $N \in \mathbb{N}$. Note that, when G acts properly discontinuously on $\mathbb{D}$, then $N = 1$ for every orbit. In this case the lemma shows that there is a neighbourhood of every point $p = q(z_o)$ homeomorphic to a neighbourhood of z_o in $\mathbb{D}$. These neighbourhoods readily give us charts that make the quotient $\mathbb{D}/G$ into a Riemann surface with $q : \mathbb{D} \to \mathbb{D}/G$ holomorphic.

Suppose that z_o has a stabilizer in G of order N. The stabilizer is a cyclic group generated by an elliptic transformation fixing z_o. Proposition 6.1.2 gives a number $\delta = \delta(p_o) > 0$ such that the discs $\mathbb{D}(w, \delta) = \{z \in \mathbb{D} : \rho(w, z) < \delta\}$ for $w \in p_o$ are either disjoint or identical. For $T \in G$ we have

$$\begin{aligned} T(\mathbb{D}(z_o, \delta)) &= \mathbb{D}(z_o, \delta) && \text{when } T \in \mathrm{Stab}(z_o)\ ; \\ T(\mathbb{D}(z_o, \delta)) \cap \mathbb{D}(z_o, \delta) &= \emptyset && \text{for } T \in G \setminus \mathrm{Stab}(z_o)\ . \end{aligned}$$

Set $\Delta = \Delta(p_o) = \{p \in \mathbb{D}/G : \widetilde{\rho}(p_o, p) < \delta\}$. Then the quotient map sends $\mathbb{D}(z_o, \delta)$ onto Δ. Each value in Δ is taken exactly N times in $\mathbb{D}(z_o, \delta)$ except for p_o, which is only taken at z_o. The points $z, z' \in \mathbb{D}(z_o, \delta)$ have $q(z) = q(z')$ if and only if $z' = T(z)$ for some $T \in \mathrm{Stab}(z_o)$.

The map

$$\beta : z \mapsto \left(\frac{z - z_o}{1 - \overline{z_o} z} \right)^N$$

maps the disc $\mathbb{D}(z_o, \delta)$ onto another disc $\mathbb{D}(0, \delta')$ (where δ' is given by $\tanh \frac{1}{2}\delta' = (\tanh \frac{1}{2}\delta)^N$). This map also has the property that $\beta(z) = \beta(z')$ if and only if $z' = T(z)$ for some $T \in \mathrm{Stab}(z_o)$. Therefore, there is an unique map

$$\phi : \Delta \to \mathbb{D}(0, \delta') \qquad \text{with} \qquad \phi(q(z)) = \left(\frac{z - z_o}{1 - \overline{z_o} z}\right)^N$$

for each $z \in \mathbb{D}(z_o, \delta)$.

Take these maps β as charts for $\mathbb{D}/G$. The transition maps are clearly Möbius transformations and so we see that the quotient is a Riemann surface with the quotient map holomorphic.

Note that the critical points of the quotient map $q : \mathbb{D} \to \mathbb{D}/G$ are those that have non-trivial stabilizers. These points are a discrete subset of $\mathbb{D}$. So, in particular, the stabilizer is trivial for all but a countable number of points in $\mathbb{D}$.

6.2 Discrete Groups

Möbius transformations in Möb($\mathbb{D}$) are represented by 2×2 complex matrices. This set of matrices has a natural Euclidean topology. Hence we can define a subgroup G of Möb($\mathbb{D}$) to be *a discrete group* if it is a discrete subset of Möb($\mathbb{D}$).

Proposition 6.2.1

A subgroup G of Möb($\mathbb{D}$) is a discrete group if and only if the identity I is isolated in G.

Proof:

For each $T \in G$ the left multiplication $L_T : G \to G;\ A \mapsto TA$ is a homeomorphism. Hence I is an isolated point of G if and only if every $T \in G$ is isolated, that is G is discrete. □

This proposition shows that a group $G \leqslant$ Möb($\mathbb{D}$) is a discrete subgroup of Möb($\mathbb{D}$) if and only if there is no sequence (T_n) of non-identity elements of G that converges to I.

The main aim of this section is to prove that discreteness and acting discontinuously on $\mathbb{D}$ are equivalent.

Theorem 6.2.2

Let G be a subgroup of Möb($\mathbb{D}$). Then G acts discontinuously on $\mathbb{D}$ if and only if G is a discrete subgroup of Möb($\mathbb{D}$).

Proof:

Suppose first that G is not a discrete subgroup of Möb($\mathbb{D}$). Then there is a sequence (T_n) of non-identity elements of G with $T_n \to I$ as $n \to \infty$. Choose any point $z_o \in \mathbb{D}$. Then $T_n(z_o) \to z_o$ as $n \to \infty$. By Proposition 6.1.1 now shows that G can not act discontinuously on $\mathbb{D}$. Thus, if G does act discontinuously on $\mathbb{D}$, then G must be a discrete subgroup of Möb($\mathbb{D}$).

For the converse, suppose that G is a discrete subgroup of Möb($\mathbb{D}$) but G does not act discontinuously on $\mathbb{D}$. By Proposition 6.1.1, there are infinitely many elements $T \in G$ with

$$\rho(0, T(0)) \leqslant r \qquad \text{for some } r < \infty\ .$$

The disc $\{z \in \mathbb{D} : \rho(0, z) \leqslant r\}$ is compact, so there is a sequence of non-identity element (T_n) in G with

$$T_n(0) \to w_o \qquad \text{as } n \to \infty$$

and w_o some point of $\mathbb{D}$ with $\rho(0, w_o) \leqslant r$.

For each $w \in \mathbb{D}$, let

$$S_w : z \mapsto \frac{z - w}{1 - \overline{w}z} \ .$$

This is a Möbius transformation and $S_w \to S_{w_o}$ as $w \to w_o$. For each $n \in \mathbb{N}$, set

$$S_n = S_{T_n(0)} \qquad \text{and} \qquad R_n = S_n \circ T_n \ .$$

Then, as $n \to \infty$, we have $S_n \to S_{w_o}$. Each R_n is a Möbius transformation of $\mathbb{D}$ that fixes 0, so it lies in the compact group of rotations

$$\{z \mapsto \omega z : |\omega| = 1\} \leqslant \text{Möb}(\mathbb{D}) \ .$$

Consequently, there is a subsequence $(R_{n'})$ that converges to some rotation R_o. Therefore,

$$T_{n'} = S_{n'}^{-1} \circ R_{n'} \to S_{w_o}^{-1} \circ R_o \qquad \text{as } n \to \infty \ .$$

This proves that G is a not a discrete subgroup of Möb($\mathbb{D}$). For, if G were discrete, then there would be an $\varepsilon > 0$ with the Euclidean distance $d(T, I) > \varepsilon$ for every $T \in G \setminus \{I\}$. This implies that $d(T, T') > \varepsilon$ for every pair of distinct elements $T, T' \in G$. Hence the sequence (T_n) from G can only converge when it is ultimately constant. □

6.3 Dirichlet Domains

Let G be a discrete group acting on $\mathbb{D}$. Then we have shown that the quotient $\mathbb{D}/G$ is a Riemann surface. To identify it, it is often useful to find a *fundamental domain* for G. This is a subdomain F of $\mathbb{D}$ such that no two points of F are in the same orbit and every orbit meets the closure $\overline{F}$ in a finite number of points. Then, we can identify the quotient $\mathbb{D}/G$ with the space obtained from $\overline{F}$ by identifying points on ∂F that lie in the same orbit. Dirichlet showed that we could always find such a fundamental domain.

Let z_o be a point $\mathbb{D}$ that has trivial stabilizer. The *Dirichlet domain with centre z_o* is the set

$$D(z_o) = \{z \in \mathbb{D} : \rho(z, z_o) < \rho(z, T(z_o)) \text{ for all } T \in G \setminus \{I\}\} \ .$$

Proposition 6.3.1 Dirichlet domains

For each point z_o with trivial stabilizer the Dirichlet domain $D(z_o)$ with centre z_o is a fundamental domain for G.

Proof:

For any two distinct points $z_o, z_1 \in \mathbb{D}$, the set

$$\gamma(z_1) = \{z \in \mathbb{D} : \rho(z, z_o) = \rho(z, z_1)\}$$

is the perpendicular bisector of the hyperbolic geodesic from z_o to z_1. Hence, it is itself a hyperbolic geodesic. The set

$$H(z_1) = \{z \in \mathbb{D} : \rho(z, z_o) < \rho(z, z_1)\}$$

is then the half-plane bounded by this geodesic that contains z_o. Each such half-plane is open, so the intersection

$$D(z_o) = \bigcap \{H(T(z_o)) : T \in G \setminus \{I\}\}$$

is an open set containing z_o.

Note that the distance from z_o to $\gamma(z_1)$ is $\frac{1}{2}\rho(z_o, z_1)$. The orbit $G(z_o)$ is a discrete subset of $\mathbb{D}$, so only finitely many points of this orbit can lie within a finite distance of any point w. This means that only finitely many of the geodesics $\gamma(T(z_o))$ meet any disc about z_o with finite hyperbolic radius. It is now simple to see that the closure of $D(z_o)$ is the intersection of the closed half-planes

$$\overline{H(z_1)} = \{z \in \mathbb{D} : \rho(z, z_o) \leqslant \rho(z, z_1)\} \ .$$

This closure is bounded by arcs of the geodesics $\gamma(T(z_o))$.

Consider a point $w \in \mathbb{D}$. The orbit $G(z_o)$ is discrete, so only finitely many points of the orbit lie within any fixed hyperbolic distance of w. Consequently, there is a point $T(z_o)$ in this orbit with $\rho(w, T(z_o))$ minimal. This means that $T^{-1}(w)$ lies in $\overline{D(z_o)}$.

Suppose another point in the orbit of w also lay in $\overline{D(z_o)}$, say $S^{-1}(w)$. Then

$$\rho(z_o, S^{-1}(w)) \leqslant \rho(S^{-1}T(z_o), S^{-1}(w)) = \rho(z_o, T^{-1}(w))$$

and so we must have equality with

$$\rho(z_o, S^{-1}(w)) = \rho(S^{-1}T(z_o), S^{-1}(w)) = \rho(z_o, T^{-1}(w)) \ .$$

Since $S^{-1}T \neq I$, we see that $S^{-1}(w)$ lies on the boundary of the Dirichlet domain $D(z_o)$. This shows that no two points of the open Dirichlet domain can lie in the same orbit.

It also shows that, if $S^{-1}(w)$ and $T^{-1}(w)$ both lie in the closure of the Dirichlet domain, then they are at the same distance from z_o. Only a finite number of points in the orbit $G(w)$ can lie at this distance from z_o. So the orbit $G(w)$ meets $\overline{D(z_o)}$ in finitely many points. $\square$