1. A continuous function $u : \Omega \to \mathbb{R}$ on a domain $\Omega \subset \mathbb{C}$ has the property S if, for each $z_o \in \Omega$, there is a $\rho > 0$ such that
$$\int_0^{2\pi} u(z_o + re^{i\theta}) \frac{d\theta}{2\pi} \geqslant u(z_o) \qquad \text{for } 0 \leqslant r < \rho.$$
Prove that
   (a) if u has property S and attains its maximum value then it is constant.
   (b) u has the property S if, and only if, u is subharmonic.
2. Let $u : \Omega \to \mathbb{R}$ be a continuous subharmonic function on a domain $\Omega \subset \mathbb{C}$. Show that, for $r \geqslant 0$, the function $\int_0^{2\pi} u(z_o + re^{i\theta})\, d\theta/2\pi$ is continuous and increasing.
Let $\phi : \mathbb{C} \to [0, \infty)$ be a smooth function with
$\phi(w) = \phi(|w|)$ for all $w \in \mathbb{C}$.
$\phi(w) = 0$ for $|w| > 1$.
$\int_{\mathbb{C}} \phi(w)\, du \wedge dv = \int_0^\infty \phi(r) 2\pi\, r dr = 1$ where $w = u + iv$.
For $\varepsilon > 0$ set $\phi_\varepsilon(w) = \varepsilon^2 \phi(w/\varepsilon)$, and $\Omega_\varepsilon = \{z \in \Omega : \mathbb{D}(z, \varepsilon) \subset \Omega\}$. Define $u_\varepsilon : \Omega_\varepsilon \to \mathbb{R}$ by
$$u_\varepsilon(z) = \int_{|w| \leqslant \varepsilon} u(z + w)\phi_\varepsilon(w)\, du \wedge dv.$$
Prove that
   (a)
$$\frac{\partial u_\varepsilon}{\partial z}(z) = \frac{\partial}{\partial z} \int_{\mathbb{C}} u(w)\phi_\varepsilon(w - z)\, du \wedge dv = -\int_{\mathbb{C}} u(w) \frac{\partial \phi_\varepsilon}{\partial z}(w - z)\, du \wedge dv.$$
   (b) u_ε is a smooth function on Ω_ε.
   (c) Each u_ε is subharmonic and $u_\varepsilon \to u$ locally uniformly as $\varepsilon \to 0$.
Thus each continuous subharmonic function is the locally uniform limit of smooth (and hence C^2) subharmonic functions.
3. Let $u : \mathbb{D} \to \mathbb{R}$ be continuous and subharmonic. Show that the least harmonic majorant of u is given by
$$\lim_{r \to 1-} \int_0^{2\pi} u(re^{i\theta}) \frac{r^2 - |z|^2}{|z - re^{i\theta}|^2} \frac{d\theta}{2\pi} = \sup_{r<1} \int_0^{2\pi} u(re^{i\theta}) \frac{r^2 - |z|^2}{|z - re^{i\theta}|^2} \frac{d\theta}{2\pi}.$$
4. Prove that a relatively compact domain with piecewise smooth boundary is regular for the Dirichlet problem.
5. Show that there cannot be a barrier at an isolated boundary point.
6. Show that every relatively compact domain Ω in R is contained within another relatively compact domain Ω' which is regular for the Dirichlet problem.
7. If g is the Green's function for R with pole at z_o and $f : S \to R$ is a conformal equivalence, then gf is the Green's function on S with pole at $f^{-1}(z_o)$.
8. Find the Green's function on the unit disc with a pole at any specified point (and prove that it is the Green's function).
9. Let g be the Green's function on a domain $\Omega \subset \mathbb{C}$ with a pole at z_o and let f be a smooth function on Ω with compact support within Ω. Prove that
$$\int_\Omega \triangle f(z) g(z)\, d\overline{z} \wedge dz = -4\pi i g(z_o).$$
Hint: Stokes' Theorem.
(This means that g defines a distribution with $\triangle g = -4\pi i \delta_{z_o}$. In partial differential equations and applied mathematics this is often taken as the definition of the Green's function.)

10. Let K be a compact subset of the non-compact Riemann surface R. Prove that we can cover K by finitely many discs so that the union of the discs is a domain Ω which is regular for the Dirichlet problem. Let g be a Green's function for Ω. Show that, for suitable small $\varepsilon > 0$, the set $\{z \in \Omega : g(z) > \varepsilon\}$ contains K and has a real analytic boundary.

    Hence every Riemann surface has a compact exhaustion by sets with real analytic boundaries.

11. Show that, for any distinct points z_o, w in any Riemann surface R there is a harmonic function $f : R \setminus \{z_o, w\} \to \mathbb{R}$ which has logarithmic singularities at z_o and w with coefficients $+1$ and -1 respectively.

12. Let R be an elliptic Riemann surface and z_o, ζ distinct points of R. Show that the function $q(z_o, \cdot) : R \setminus \{z_o, \zeta\} \to \mathbb{R}$ on the parabolic surface $R \setminus \{\zeta\}$ has a logarithmic singularity at ζ with coefficient -1 (and at z_o with coefficient $+1$).

13. Show that every Riemann surface is triangulable.